Canonical bases and affine Hecke algebras of type B

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Abstract. We prove a series of conjectures of Enomoto and Kashiwara on canonical bases and branching rules of affine Hecke algebras of type B. The main ingredient of the proof is a new graded Ext-algebra associated with quiver with involutions that we compute explicitly.

Introduction

A new family of graded algebras, called KLR algebras, has been recently introduced in [KL], [R]. These algebras yield a categorification of $\mathbf{f}$, the negative part of the quantized enveloping algebra of any type. In particular, one can obtain a new interpretation of the canonical bases, see [VV]. In type A or $\mathrm{A}^{(1)}$ the KLR algebras are Morita equivalent to the affine Hecke algebras and their cyclotomic quotients. Hence they give a new way to understand the categorification of the simple highest weight modules and the categorification of $\mathbf{f}$ via some Hecke algebras of type A or $\mathrm{A}^{(1)}$. See [BK] for instance. One of the advantages of KLR algebras is that they are graded, while the affine Hecke algebras are not. This explain why KLR algebras are better adapted than affine Hecke algebras to describe canonical bases. Indeed one could view KLR algebras as an intermediate object between the representation theory of affine Hecke algebras and its Kazhdan-Lusztig geometric counterpart in term of perverse sheaves. This is central in [VV], where KLR algebras are proved to be isomorphic to the Ext-algebras of some complex of constructible sheaves.

In the other hand, the (branching rules for) affine Hecke algebras of type B have been investigated quite recently, see [E], [EK1,2,3], [Ka], [M]. Lusztig's description of the canonical basis of $\mathbf{f}$ in type $\mathrm{A}^{(1)}$ in [L1] implies that this basis can be naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type A. This identification was mentioned in [G], and was used in [A]. More precisely, there is a linear isomorphism between $\mathbf{f}$ and the Grothendieck group of finite dimensional modules of the affine Hecke algebras of type A, and it is proved in [A] that the induction/restriction functors for affine Hecke algebras are given by the action of the Chevalley generators and their transposed operators with respect to some symmetric bilinear form on $\mathbf{f}$. In [E], [EK1,2,3] a similar behavior is conjectured and studied for affine Hecke algebras of type B. Here $\mathbf{f}$ is replaced by an explicit module ${}^\theta\mathbf{V}(\lambda)$ over an explicit algebra ${}^\theta\mathbf{B}$. First, it is conjectured that ${}^\theta\mathbf{V}(\lambda)$ admits a canonical basis. Next, it is conjectured that this basis is naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type B. Further, in this identification the branching rules of the affine Hecke algebras of type B are

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given by the ${}^\theta\mathbf{B}$-action on ${}^\theta\mathbf{V}(\lambda)$. The first conjecture has been proved in [E] under the restrictive assumption that $\lambda = 0$. Here we prove the whole set of conjectures. Indeed, our construction is slightly more general, see the appendix.

Roughtly speaking our argument is as follows. In [E] a geometric description of the canonical basis of ${}^\theta\mathbf{V}(0)$ was given. This description is similar to Lusztig's description of the canonical basis of $\mathbf{f}$ via perverse sheaves on the moduli stack of representations of some quiver. It is given in terms of perverse sheaves on the moduli stack of representations of a quiver with involution. First we give a analogue of this for ${}^\theta\mathbf{V}(\lambda)$ for any λ. This yields the existence of a canonical basis ${}^\theta\mathbf{G}^{\text{low}}(\lambda)$ for ${}^\theta\mathbf{V}(\lambda)$ for arbitrary λ. Then we compute explicitely the Ext-algebras between complexes of constructible sheaves naturally attached to quivers with involutions. These complexes enter in a natural way in the definition of ${}^\theta\mathbf{G}^{\text{low}}(\lambda)$. This computation yields a new family of graded algebras ${}^\theta\mathbf{R}_m$ where m is a nonnegative integer. We prove that the algebras ${}^\theta\mathbf{R}_m$ are Morita equivalent to the affine Hecke algebras of type B. Finally we describe ${}^\theta\mathbf{V}(\lambda)$ and the basis ${}^\theta\mathbf{G}^{\text{low}}(\lambda)$ in terms of the Grothendieck group of ${}^\theta\mathbf{R}_m$.

The plan of the paper is the following. Section 1 contains some basic notation for Lusztig's theory of perverse sheaves on the moduli stack of representations of quivers. Section 2 yields similar notation for the case of quivers with involutions. Our setting is more general than in [E], where only the case $\lambda = 0$ is considered. In Section 3 we introduce the convolution algebra associated with a quiver with involution. The main result of Section 4 is Theorem 4.17 where the polynomial representation of the Ext-algebra $\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$ associated with a quiver with involution is computed. Here $\mathbf{\Lambda}$ is a I-graded $\mathbb{C}$-vector space of dimension vector $\lambda \in \mathbb{N}I$, while $\mathbf{V}$ is a I-graded $\mathbb{C}$-vector space with a non-degenerate symmetric bilinear form of dimension vector $\nu \in \mathbb{N}I$. The polynomial representation of $\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$ is faithful. In Section 5 we give the main properties of the graded algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. In Section 6 we introduce the affine Hecke algebra of type B and we prove that it is Morita equivalent to ${}^\theta\mathbf{R}_m$, a specialization of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. Section 7 is a reminder on KLR algebras and on the main result of [VV]. In Section 8 we categorify the module ${}^\theta\mathbf{V}(\lambda)$ from [EK1] using the graded algebra ${}^\theta\mathbf{R}_m$. In Section 9 we prove the isomorphism ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} = \mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$. This is essential to compare the construction from Section 8 with that in Section 10. In Section 10 we give a categorification of ${}^\theta\mathbf{V}(\lambda)$ "à la Lusztig" in terms of perverse sheaves on the moduli stack of representations of quivers with involution. This is essentially the same construction as in [E]. However, since we need a more general setting than in loc. cit. we have briefly reproduced the main steps of the construction. One of our initial motivations was to give a completely algebraic proof of the conjectures, without any perverse sheaves at all. We still do not know how to do this. The main result of the paper is Theorem 10.19.

The same technic yields similar results for affine Hecke algebras of any classical type. The case of type D is done in [SVV], the case of type C is done in the appendix. Note that the idea to use canonical bases technics to study affine Hecke algebras in non A type is not new, see [L3], [L4]. At the moment we do not know the precise relation between loc. cit. and our approach.

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Contents

0. Notation

0.1. Combinatorics. Given a positive integer m and a tuple $\mathbf{m} = (m_1, m_2, \dots m_r)$ of positive integers we write $\mathfrak{S}_m$ for the symmetric group and $\mathfrak{S}_\mathbf{m}$ for the group $\prod_{l=1}^r \mathfrak{S}_{m_l}$. Set

$$|\mathbf{m}| = \sum_{l=1}^r m_l, \quad \ell_\mathbf{m} = \sum_{l=1}^r \ell_{m_l}, \quad \ell_m = m(m-1)/2.$$

We use the following notation for v-numbers

$$\langle m\rangle = \sum_{l=1}^m v^{m+1-2l}, \quad \langle m\rangle! = \prod_{l=1}^m \langle l\rangle, \quad \left\langle \begin{matrix} m+n \\ n \end{matrix} \right\rangle = \frac{\langle m+n\rangle!}{\langle m\rangle!\langle n\rangle!}, \quad \langle \mathbf{m}\rangle! = \prod_{l=1}^r \langle m_l\rangle!.$$

Given two tuples $\mathbf{m} = (m_1, m_2, \dots m_r)$, $\mathbf{m}' = (m'_1, m'_2, \dots m'_{r'})$ we define the tuple

$$\mathbf{m}\mathbf{m}' = (m_1, m_2, \dots m_r, m'_1, m'_2, \dots m'_{r'}).$$

0.2. Graded modules over graded algebras. Let $\mathbf{k}$ be an algebraically closed field of caracteristic 0. Let $\mathbf{R} = \bigoplus_d \mathbf{R}_d$ be a graded $\mathbf{k}$-algebra. Unless specified otherwise the word graded we'll always mean $\mathbb{Z}$-graded. Let **R-mod** be the category of finitely generated graded $\mathbf{R}$-modules, **R-fmod** be the full subcategory of finite-dimensional graded modules and **R-proj** be the full subcategory of projective objects. Unless specified otherwise a module is always a left module. We'll abbreviate

$$K(\mathbf{R}) = [\mathbf{R}\text{-}\mathbf{proj}], \quad G(\mathbf{R}) = [\mathbf{R}\text{-}\mathbf{fmod}].$$

Here $[\mathcal{C}]$ denotes the Grothendieck group of an exact category $\mathcal{C}$. Assume that the $\mathbf{k}$-vector spaces $\mathbf{R}_d$ are finite dimensional for each d. Then $K(\mathbf{R})$ is a free Abelian group with a basis formed by the isomorphism classes of the indecomposable objects in **R-proj**, and $G(\mathbf{R})$ is a free Abelian group with a basis formed by the isomorphism classes of the simple objects in **R-fmod**. Given an object M of **R-proj** or **R-fmod** let $[M]$ denote its class in $K(\mathbf{R})$, $G(\mathbf{R})$ respectively. When there is no risk of confusion we abbreviate $M = [M]$. We'll write $[M : N]$ for the composition multiplicity of the $\mathbf{R}$-module N in the $\mathbf{R}$-module M. Consider the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. If the grading of $\mathbf{R}$ is bounded below then the $\mathcal{A}$-modules $K(\mathbf{R})$, $G(\mathbf{R})$ are free. Here $\mathcal{A}$ acts on $G(\mathbf{R})$, $K(\mathbf{R})$ as follows

$$vM = M[1], \quad v^{-1}M = M[-1].$$

For any M, N in **R-mod** let

$$\hom_{\mathbf{R}}(M, N) = \bigoplus_d \mathrm{Hom}_{\mathbf{R}}(M, N[d])$$

be the graded $\mathbf{k}$-vector space of all $\mathbf{R}$-module homomorphisms. If $\mathbf{R} = \mathbf{k}$ we'll omit the subscript $\mathbf{R}$ in hom's and in tensor products. As much as possible we'll use the following convention : graded objects are denoted by minuscules and non-graded ones by majuscules. In particular **R-Mod** will denote the category of finitely generated (non-graded) $\mathbf{R}$-modules. We 'll abbreviate

$$\mathrm{Hom} = \mathrm{Hom}_{\mathbf{k}}, \quad \otimes = \otimes_{\mathbf{k}}, \quad etc.$$

For a graded $\mathbf{k}$-vector space $M = \bigoplus_d M_d$ we'll write

$$\mathrm{gdim}(M) = \sum_d v^d \dim(M_d),$$

where dim is the dimension over $\mathbf{k}$.

0.3. Constructible sheaves. Given an action of a complex linear algebraic group G on a quasiprojective algebraic variety X over $\mathbb{C}$ we write $\boldsymbol{\mathcal{D}}_G(X)$ for the bounded derived category of complexes of G-equivariant sheaves of $\mathbf{k}$-vector spaces on X. Objects of $\boldsymbol{\mathcal{D}}_G(X)$ are referred to as complexes. If $G = \{e\}$, the trivial group, we abbreviate $\boldsymbol{\mathcal{D}}(X) = \boldsymbol{\mathcal{D}}_G(X)$. For each complexes $\mathcal{L}$, $\mathcal{L}'$ we'll abbreviate

$$\mathrm{Ext}^*_G(\mathcal{L}, \mathcal{L}') = \mathrm{Ext}^*_{\boldsymbol{\mathcal{D}}_G(X)}(\mathcal{L}, \mathcal{L}'), \quad \mathrm{Ext}^*(\mathcal{L}, \mathcal{L}') = \mathrm{Ext}^*_{\boldsymbol{\mathcal{D}}(X)}(\mathcal{L}, \mathcal{L}')$$

if no confusion is possible. The constant sheaf on X with stalk $\mathbf{k}$ will be denoted $\mathbf{k}$. For any object $\mathcal{L}$ of $\boldsymbol{\mathcal{D}}_G(X)$ let $H^*_G(X, \mathcal{L})$ be the space of G-equivariant cohomology with coefficients in $\mathcal{L}$. Let $\mathcal{D} \in \boldsymbol{\mathcal{D}}_G(X)$ be the G-equivariant dualizing complex, see [BL, def. 3.5.1]. For each $\mathcal{L}$ let

$$\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{D})$$

be its Verdier dual, where $\mathcal{H}om$ is the internal Hom. Recall that

$$(\mathcal{L}^\vee)^\vee = \mathcal{L}, \quad \mathrm{Ext}^*_G(\mathcal{L}, \mathcal{D}) = H^*_G(X, \mathcal{L}^\vee), \quad \mathrm{Ext}^*_G(\mathbf{k}, \mathcal{L}) = H^*_G(X, \mathcal{L}).$$

We define the space of G-equivariant homology by

$$H_*^G(X,\mathbf{k}) = H_G^*(X,\mathcal{D}).$$

Note that $\mathcal{D} = \mathbf{k}[2d]$ if X is a smooth G-variety of pure dimension d. Consider the following graded $\mathbf{k}$-algebra

$$\mathbf{S}_G = H_G^*(\bullet,\mathbf{k}).$$

The graded $\mathbf{k}$-vector space $H_*^G(X,\mathbf{k})$ has a natural structure of a graded $\mathbf{S}_G$-module. We have

$$H_*^G(\bullet,\mathbf{k}) = \mathbf{S}_G$$

as graded $\mathbf{S}_G$-module. There is a canonical graded $\mathbf{k}$-algebra isomorphism

$$\mathbf{S}_G \simeq \mathbf{k}[\mathfrak{g}]^G.$$

Here the symbol $\mathfrak{g}$ denotes the Lie algebra of G and a G-invariant homogeneous polynomial over $\mathfrak{g}$ of degree d is given the degree $2d$ in $\mathbf{S}_G$.

Fix a morphism of quasi-projective algebraic G-varieties $f : X \to Y$. If f is a proper map there is a direct image homomorphism

$$f_* : H_*^G(X,\mathbf{k}) \to H_*^G(Y,\mathbf{k}).$$

If f is a smooth map of relative dimension d there is an inverse image homomorphism

$$f^* : H_i^G(Y,\mathbf{k}) \to H_{i-2d}^G(X,\mathbf{k}), \quad \forall i.$$

If X has pure dimension d there is a natural homomorphism

$$H_G^i(X,\mathbf{k}) \to H_{i-2d}^G(X,\mathbf{k}).$$

It is invertible if X is smooth. The image of the unit is called the fundamental class of X in $H_*^G(X,\mathbf{k})$. We denote it by $[X]$. If $f : X \to Y$ is the embedding of a G-stable closed subset and $X' \subset X$ is the union of the irreducible components of maximal dimension then the image of $[X']$ by the map f_* is the fundamental class of X in $H_*^G(Y,\mathbf{k})$. It is again denoted by $[X]$.

1. Reminder on quivers and extensions

1.1. Representations of quivers. We assume given a nonempty quiver Γ such that no arrow may join a vertex to itself. Recall that Γ is a tuple $(I, H, h \mapsto h', h \mapsto h'')$ where I is the set of vertices, H is the set of arrows, and for $h \in H$ the vertices $h', h'' \in I$ are the origin and the goal of h respectively. Note that the set I may be infinite. For $i, j \in I$ we write

$$H_{i,j} = \{h \in H; h' = i, h'' = j\}.$$

We'll abbreviate $i \to j$ for $H_{i,j} \neq \emptyset$, $i \not\to j$ for $H_{i,j} = \emptyset$, and $h : i \to j$ for $h \in H_{i,j}$. Let $h_{i,j}$ be the number of elements in $H_{i,j}$ and set

$$i \cdot j = -h_{i,j} - h_{j,i}, \quad i \cdot i = 2, \quad i \neq j.$$

Let $\boldsymbol{\mathcal{V}}$ be the category of finite-dimensional I-graded $\mathbb{C}$-vector spaces $\mathbf{V} = \bigoplus_{i \in I} \mathbf{V}_i$ with morphisms being linear maps respecting the grading. For $\nu = \sum_i \nu_i i$ in $\mathbb{N}I$ let $\boldsymbol{\mathcal{V}}_\nu$ be the full subcategory of $\boldsymbol{\mathcal{V}}$ whose objects are those $\mathbf{V}$ such that $\dim(\mathbf{V}_i) = \nu_i$ for all i. We call ν the dimension vector of $\mathbf{V}$. Given an object $\mathbf{V}$ of $\boldsymbol{\mathcal{V}}$ let

$$E_{\mathbf{V}} = \bigoplus_{h \in H} \mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''}).$$

The algebraic group $G_{\mathbf{V}} = \prod_i GL(\mathbf{V}_i)$ acts on $E_{\mathbf{V}}$ by $(g, x) \mapsto gx = y$ where $y_h = g_{h''} x_h g_{h'}^{-1}$, $g = (g_i)$, $x = (x_h)$, and $y = (y_h)$.

Fix a nonzero element ν of $\mathbb{N}I$. Let Y^ν be the set of all pairs $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ where $\mathbf{i} = (i_1, i_2, \dots i_k)$ is a sequence of elements of I and $\mathbf{a} = (a_1, a_2, \dots a_k)$ is a sequence of positive integers such that $\sum_l a_l \, i_l = \nu$. Note that the assignment

$$\mathbf{y} \mapsto (a_1 i_1, a_2 i_2, \dots a_k i_k) \tag{1.1}$$

identifies Y^ν with a set of sequences

$$\nu^1, \nu^2, \dots, \nu^k \in \mathbb{N}I \quad \text{with } \nu = \sum_{l=1}^{k} \nu^l. \tag{1.2}$$

For each pair $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ as above we'll call $\mathbf{a}$ the multiplicity of $\mathbf{y}$. Let $I^\nu \subset Y^\nu$ be the set of all pairs $\mathbf{y}$ with multiplicity $(1, 1, \dots, 1)$. We'll abbreviate $\mathbf{i}$ for a pair $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ which lies in I^ν. Given a positive integer m we have $\bigsqcup_\nu I^\nu = I^m$, where ν runs over the set of elements ν of $\mathbb{N}I$ with $|\nu| = m$. Here, we write $\nu = \sum_{i \in I} \nu_i i$ and $|\nu| = \sum_i \nu_i$. In a similar way, we define $Y^m = \bigsqcup_\nu Y^\nu$.

1.2. Flags. Let $\nu \in \mathbb{N}I$, $\nu \neq 0$, and assume that $\mathbf{V}$ lies in $\boldsymbol{\mathcal{V}}_\nu$. For each sequence $\mathbf{y} = (\nu^1, \nu^2, \dots \nu^k)$ as in (1.1), (1.2), a flag of type $\mathbf{y}$ in $\mathbf{V}$ is a sequence

$$\phi = (\mathbf{V} = \mathbf{V}^0 \supset \mathbf{V}^1 \supset \cdots \supset \mathbf{V}^k = 0)$$

of I-graded subspace of $\mathbf{V}$ such that for any l the I-graded subspace $\mathbf{V}^{l-1}/\mathbf{V}^l$ belongs to $\boldsymbol{\mathcal{V}}_{\nu^l}$. Let $F_{\mathbf{V},\mathbf{y}}$ be the variety of all flags of type $\mathbf{y}$ in $\mathbf{V}$. The group $G_{\mathbf{V}}$ acts transitively on $F_{\mathbf{V},\mathbf{y}}$ in the obvious way, yielding a smooth projective $G_{\mathbf{V}}$-variety structure on $F_{\mathbf{V},\mathbf{y}}$.

If $x \in E_{\mathbf{V}}$ we say that the flag ϕ is x-stable if $x_h(\mathbf{V}^l_{h'}) \subset \mathbf{V}^l_{h''}$ for all h, l. Let $\widetilde{F}_{\mathbf{V},\mathbf{y}}$ be the variety of all pairs (x, ϕ) such that ϕ is x-stable. Set $d_{\mathbf{y}} = \dim(\widetilde{F}_{\mathbf{V},\mathbf{y}})$. The group $G_{\mathbf{V}}$ acts on $\widetilde{F}_{\mathbf{V},\mathbf{y}}$ by $g : (x, \phi) \mapsto (gx, g\phi)$. The first projection gives a $G_{\mathbf{V}}$-equivariant proper morphism

$$\pi_{\mathbf{y}} : \widetilde{F}_{\mathbf{V},\mathbf{y}} \to E_{\mathbf{V}}.$$

1.3. Ext-algebras. Let $\nu \in \mathbb{N}I$, $\nu \neq 0$, and assume that $\mathbf{V} \in \boldsymbol{\mathcal{V}}_\nu$. We abbreviate $\mathbf{S}_\mathbf{V} = \mathbf{S}_{G_\mathbf{V}}$. For each sequence $\mathbf{y} \in Y^\nu$ we have the following semisimple complexes in $\boldsymbol{\mathcal{D}}_{G_\mathbf{V}}(E_\mathbf{V})$

$$\mathcal{L}_\mathbf{y} = (\pi_\mathbf{y})_!(\mathbf{k}), \quad \mathcal{L}_\mathbf{y}^\vee = \mathcal{L}_\mathbf{y}[2d_\mathbf{y}], \quad \mathcal{L}_\mathbf{y}^\delta = \mathcal{L}_\mathbf{y}[d_\mathbf{y}].$$

For $\mathbf{y}, \mathbf{y}'$ in Y^ν we consider the graded $\mathbf{S}_\mathbf{V}$-module

$$\mathbf{Z}_{\mathbf{V},\mathbf{y},\mathbf{y}'} = \mathrm{Ext}^*_{G_\mathbf{V}}(\mathcal{L}_\mathbf{y}^\vee, \mathcal{L}_{\mathbf{y}'}^\vee).$$

For $\mathbf{y}, \mathbf{y}', \mathbf{y}''$ in Y^ν the Yoneda composition is a homogeneous $\mathbf{S}_\mathbf{V}$-bilinear map of degree zero

$$\star : \mathbf{Z}_{\mathbf{V},\mathbf{y},\mathbf{y}'} \times \mathbf{Z}_{\mathbf{V},\mathbf{y}',\mathbf{y}''} \to \mathbf{Z}_{\mathbf{V},\mathbf{y},\mathbf{y}''}.$$

The map $\star$ equips the graded $\mathbf{k}$-vector space

$$\mathbf{Z}_\mathbf{V} = \bigoplus_{\mathbf{i},\mathbf{i}' \in I^\nu} \mathbf{Z}_{\mathbf{V},\mathbf{i},\mathbf{i}'}$$

with the structure of an associative graded $\mathbf{S}_\mathbf{V}$-algebra with 1. If there is no ambiguity we'll omit the symbol $\star$. We set

$$\mathbf{F}_{\mathbf{V},\mathbf{y}} = \mathrm{Ext}^*_{G_\mathbf{V}}(\mathcal{L}_\mathbf{y}^\vee, \mathcal{D}), \quad \mathbf{F}_\mathbf{V} = \bigoplus_{\mathbf{i} \in I^\nu} \mathbf{F}_{\mathbf{V},\mathbf{i}}.$$

For $\mathbf{y}, \mathbf{y}'$ in Y^ν the Yoneda product gives a graded $\mathbf{S}_\mathbf{V}$-bilinear map $\mathbf{Z}_{\mathbf{V},\mathbf{y},\mathbf{y}'} \times \mathbf{F}_{\mathbf{V},\mathbf{y}'} \to \mathbf{F}_{\mathbf{V},\mathbf{y}}$. This yields a left graded representation of $\mathbf{Z}_\mathbf{V}$ on $\mathbf{F}_\mathbf{V}$. For each $\mathbf{i} \in I^\nu$ let $1_{\mathbf{V},\mathbf{i}} \in \mathbf{Z}_{\mathbf{V},\mathbf{i},\mathbf{i}}$ denote the identity of $\mathcal{L}_\mathbf{i}$. The elements $1_{\mathbf{V},\mathbf{i}}$ form a complete set of orthogonal idempotents of $\mathbf{Z}_\mathbf{V}$ such that

$$\mathbf{Z}_{\mathbf{V},\mathbf{i},\mathbf{i}'} = 1_{\mathbf{V},\mathbf{i}} \star \mathbf{Z}_\mathbf{V} \star 1_{\mathbf{V},\mathbf{i}'}, \quad \mathbf{F}_{\mathbf{V},\mathbf{i}} = 1_{\mathbf{V},\mathbf{i}} \star \mathbf{F}_\mathbf{V}.$$

We'll change the grading of $\mathbf{Z}_\mathbf{V}$ in the following way. Put

$$\mathbf{Z}^\delta_{\mathbf{V},\mathbf{i},\mathbf{i}'} = \mathrm{Ext}^*_{G_\mathbf{V}}(\mathcal{L}_\mathbf{i}^\delta, \mathcal{L}_{\mathbf{i}'}^\delta), \quad \mathbf{Z}^\delta_\mathbf{V} = \bigoplus_{\mathbf{i},\mathbf{i}' \in I^\nu} \mathbf{Z}^\delta_{\mathbf{V},\mathbf{i},\mathbf{i}'}.$$

The graded $\mathbf{k}$-algebra $\mathbf{Z}^\delta_\mathbf{V}$ depends only on the dimension vector of $\mathbf{V}$. We'll write

$$\mathbf{R}(\Gamma)_\nu = \mathbf{Z}^\delta_\mathbf{V}.$$

This graded $\mathbf{k}$-algebra has been computed explicitely in [VV]. The same result has also been anounced by R. Rouquier. See Section 7 for more details. We set also $I^0 = \{\emptyset\}$, $\mathcal{L}^\delta_\emptyset = \mathbf{k}$ (the constant sheaf over $\{0\}$)

$$\mathbf{R}(\Gamma)_0 = \mathbf{Z}^\delta_{\{0\}} = \mathbf{k}.$$

2. Quivers with involutions

In this section we introduce an analogue of the Ext-algebra $\mathbf{R}(\Gamma)_\nu$. It is associated with a quiver with an involution.

2.1. Representations of quivers with involution. Fix a nonempty quiver Γ such that no arrow may join a vertex to itself. An involution θ on Γ is a pair of involutions on I and H, both denoted by θ, such that the following properties hold for $h \in H$

- $\theta(h)' = \theta(h'')$ and $\theta(h)'' = \theta(h')$,
- $\theta(h') = h''$ iff $\theta(h) = h$.

We'll always assume that θ has no fixed points in I, i.e., there is no $i \in I$ such that $\theta(i) = i$. To simplify we'll say that θ has no fixed points.

Let ${}^\theta\boldsymbol{\mathcal{V}}$ be the category of finite-dimensional I-graded $\mathbb{C}$-vector spaces $\mathbf{V}$ with a non-degenerate symmetric bilinear form ϖ such that

$$(\mathbf{V}_i)^\perp = \bigoplus_{j \neq \theta(i)} \mathbf{V}_j.$$

To simplify we'll say that $\mathbf{V}$ belongs to ${}^\theta\boldsymbol{\mathcal{V}}$ if there is a bilinear form ϖ such that the pair $(\mathbf{V}, \varpi)$ lies in ${}^\theta\boldsymbol{\mathcal{V}}$. The morphisms in ${}^\theta\boldsymbol{\mathcal{V}}$ are the linear maps which respect the grading and the bilinear form. Let

$${}^\theta\mathbb{N}I = \{\nu = \textstyle\sum_i \nu_i\, i \in \mathbb{N}I; \nu_{\theta(i)} = \nu_i, \forall i\}.$$

For $\nu \in {}^\theta\mathbb{N}I$ let ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ be the full subcategory of ${}^\theta\boldsymbol{\mathcal{V}}$ consisting of the pairs $(\mathbf{V}, \varpi)$ such that $\mathbf{V}$ lies in $\boldsymbol{\mathcal{V}}_\nu$. Note that $|\nu|$ is an even integer. We'll usually write $|\nu| = 2m$ with $m \in \mathbb{N}$. Given $\mathbf{V}$ in ${}^\theta\boldsymbol{\mathcal{V}}$ and $\boldsymbol{\Lambda}$ in $\boldsymbol{\mathcal{V}}$ we let

$${}^\theta E_{\mathbf{V}} = \{x = (x_h) \in E_{\mathbf{V}}; x_{\theta(h)} = -{}^t x_h,\ \forall h \in H\},$$

$${}^\theta G_{\mathbf{V}} = \{g \in G_{\mathbf{V}}; g_{\theta(i)} = {}^t g_i^{-1}, \forall i \in I\},$$

$${}^\theta E_{\boldsymbol{\Lambda},\mathbf{V}} = {}^\theta E_{\mathbf{V}} \times L_{\boldsymbol{\Lambda},\mathbf{V}}, \quad L_{\boldsymbol{\Lambda},\mathbf{V}} = \mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\boldsymbol{\Lambda}, \mathbf{V}).$$

The algebraic groups ${}^\theta G_{\mathbf{V}}$, $G_{\boldsymbol{\Lambda}}$ act on ${}^\theta E_{\mathbf{V}}$, $L_{\boldsymbol{\Lambda},\mathbf{V}}$ in the obvious way.

2.2. Generalities on isotropic flags. Given a finite dimensional $\mathbb{C}$-vector space $\mathbf{W}$ with a non-degenerate symmetric bilinear form ϖ, an *isotropic flag in* $\mathbf{W}$ is a sequence of subspaces

$$\phi = (\mathbf{W} = \mathbf{W}^{-k} \supset \mathbf{W}^{1-k} \supset \cdots \supset \mathbf{W}^k = 0)$$

such that $(\mathbf{W}^l)^\perp = \mathbf{W}^{-l}$ for any $l = -k, 1-k, \ldots, k-1, k$. Here the symbol $\perp$ means the orthogonal relative to ϖ. In particular $\mathbf{W}^0$ is a Lagrangian subspace of $\mathbf{W}$. Let $F(\mathbf{W})$ be the variety of all complete flags in $\mathbf{W}$, and $F(\mathbf{W}, \varpi)$ be the subvariety of all complete isotropic flags, i.e., we require that $\phi = (\mathbf{W}^l)$ is an isotropic flag such that $\mathbf{W}^l$ has the dimension $m - l$ and $k = m$. If $\mathbf{W}$ has dimension $2m$ then $F(\mathbf{W}, \varpi)$ has dimension $2\ell_m = m(m-1)$.

2.3. Sequences. Fix a nonzero dimension vector ν in ${}^\theta\mathbb{N}I$. Let ${}^\theta Y^\nu$ be the set of all pairs $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ in Y^ν such that

$$\mathbf{i} = (i_{1-k}, \dots, i_{k-1}, i_k), \quad \mathbf{a} = (a_{1-k}, \dots, a_{k-1}, a_k), \quad \theta(i_l) = i_{1-l}, \quad a_l = a_{1-l}.$$

As in (1.1) we can identify a pair $\mathbf{y}$ as above with a sequence

$$\nu^{1-k}, \dots, \nu^{k-1}, \nu^k \in \mathbb{N}I, \quad \theta(\nu^l) = \nu^{1-l}, \quad \sum_l \nu^l = \nu.$$

Let ${}^\theta I^\nu \subset {}^\theta Y^\nu$ be the set of all pairs $\mathbf{y}$ of multiplicity $(1, 1, \dots, 1)$. We'll abbreviate $\mathbf{i} = (\mathbf{i}, \mathbf{a})$ for each pair in ${}^\theta I^\nu$. Note that a sequence in ${}^\theta I^\nu$ contains $|\nu| = 2m$ terms. Unless specified otherwise the entries of a sequence $\mathbf{i}$ in ${}^\theta I^\nu$ will always denoted by

$$\mathbf{i} = (i_{1-m}, \dots, i_{m-1}, i_m).$$

Finally, we set

$${}^\theta I^m = \bigcup_\nu {}^\theta I^\nu, \quad \nu \in {}^\theta\mathbb{N}I, \quad |\nu| = 2m,$$

and we define ${}^\theta Y^m$ in the same way.

2.4. Definition of the map ${}^\theta\pi_{\mathbf{\Lambda},\mathbf{y}}$. Fix $\nu \in {}^\theta\mathbb{N}I$, $\nu \neq 0$, and $\lambda \in \mathbb{N}I$. Fix an object $\mathbf{V}$ in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ and an object $\mathbf{\Lambda}$ in $\boldsymbol{\mathcal{V}}_\lambda$. For $\mathbf{y}$ in ${}^\theta Y^\nu$ an *isotropic flag of type* $\mathbf{y}$ in $\mathbf{V}$ is an isotropic flag

$$\phi = (\mathbf{V} = \mathbf{V}^{-k} \supset \mathbf{V}^{1-k} \supset \cdots \supset \mathbf{V}^k = 0)$$

such that $\mathbf{V}^{l-1}/\mathbf{V}^l$ lies in $\boldsymbol{\mathcal{V}}_{\nu^l}$ for each l. We define ${}^\theta F_{\mathbf{V},\mathbf{y}}$ to be the variety of all isotropic flags of type $\mathbf{y}$ in $\mathbf{V}$. Next, we define ${}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y}}$ to be the variety of all tuples (x, y, ϕ) satisfying the following conditions :

- $x \in {}^\theta E_{\mathbf{V}}$ and $\phi \in {}^\theta F_{\mathbf{V},\mathbf{y}}$ is *stable by* x, i.e., $x(\mathbf{V}^l) \subset \mathbf{V}^l$ for each l,
- $y \in L_{\mathbf{\Lambda},\mathbf{V}}$ and $y(\mathbf{\Lambda}) \subset \mathbf{V}^0$.

We set

$$d_{\lambda,\mathbf{y}} = \dim({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y}}).$$

We have the following formulas.

2.5. Proposition. *For* $\mathbf{i} \in {}^\theta I^\nu$ *we have*

(a) $\dim({}^\theta F_{\mathbf{V},\mathbf{i}}) = \ell_\nu/2$,

(b) $d_{\lambda,\mathbf{i}} = \ell_\nu/2 + \sum_{k<l;\, k+l\neq 1} h_{i_k,i_l}/2 + \sum_{1\leqslant l} \lambda_{i_l}$.

Proof : Fix a subset $J \subset I$ such that $I = J \sqcup \theta(J)$. Set $\mathbf{V}_J = \bigoplus_{j\in J} \mathbf{V}_j$. The assignment $(\mathbf{V}^k) \mapsto (\mathbf{V}^k \cap \mathbf{V}_J)$ takes ${}^\theta F_{\mathbf{V},\mathbf{i}}$ isomorphically onto

$$\prod_{j\in J} F(\mathbf{V}_j).$$

Thus we have

$$\dim({}^\theta F_{\mathbf{V},\mathbf{i}}) = \sum_{j\in J} \ell_{\nu_j} = \sum_{i\in I} \ell_{\nu_i}/2.$$

Next, fix a sequence $\mathbf{i}$ as above and fix a flag $\phi = (\mathbf{V}^k)$ in ${}^\theta F_{\mathbf{V},\mathbf{i}}$. Then we have

$$d_{\lambda,\mathbf{i}} = \ell_\nu/2 + \dim\{x \in {}^\theta E_{\mathbf{V}};\, x(\mathbf{V}^k) \subset \mathbf{V}^k,\, \forall k\} + \dim\{y \in L_{\mathbf{\Lambda},\mathbf{V}};\, y(\mathbf{\Lambda}) \subset \mathbf{V}^0\}.$$

Finally we have (see the discussion in Section 4.9)

$$\dim\{x \in {}^\theta E_{\mathbf{V}};\, x(\mathbf{V}^k) \subset \mathbf{V}^k,\, \forall k\} = \sum_{k<l;\, k+l\neq 1} h_{i_k,i_l}/2,$$

$$\dim\{y \in L_{\mathbf{\Lambda},\mathbf{V}};\, y(\mathbf{\Lambda}) \subset \mathbf{V}^0\} = \sum_{1\leqslant l\leqslant m} \lambda_{i_l}.$$

□

The group ${}^\theta G_{\mathbf{V}}$ acts transitively on ${}^\theta F_{\mathbf{V},\mathbf{y}}$. It acts also on ${}^\theta \widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y}}$. The first projection gives a ${}^\theta G_{\mathbf{V}}$-equivariant proper morphism

$${}^\theta \pi_{\mathbf{\Lambda},\mathbf{y}} : {}^\theta \widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y}} \to {}^\theta E_{\mathbf{\Lambda},\mathbf{V}}.$$

For a future use we introduce also the obvious projection

$$p : {}^\theta \widetilde{F}_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta F_{\mathbf{V}}, \quad {}^\theta \widetilde{F}_{\mathbf{\Lambda},\mathbf{V}} = \coprod_{\mathbf{i}\in {}^\theta I^\nu} {}^\theta \widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, \quad {}^\theta F_{\mathbf{V}} = \coprod_{\mathbf{i}\in {}^\theta I^\nu} {}^\theta F_{\mathbf{V},\mathbf{i}}.$$

2.6. Ext-algebras. Let λ, ν, $\mathbf{\Lambda}$, $\mathbf{V}$ be as above. We abbreviate ${}^\theta \mathbf{S}_{\mathbf{V}} = \mathbf{S}_{{}^\theta G_{\mathbf{V}}}$. For $\mathbf{y} \in {}^\theta Y^\nu$ we define the following semisimple complexes in $\boldsymbol{\mathcal{D}}_{{}^\theta G_{\mathbf{V}}}({}^\theta E_{\mathbf{\Lambda},\mathbf{V}})$

$${}^\theta \mathcal{L}_{\mathbf{y}} = ({}^\theta \pi_{\mathbf{\Lambda},\mathbf{y}})_!(\mathbf{k}), \quad {}^\theta \mathcal{L}^\vee_{\mathbf{y}} = {}^\theta \mathcal{L}_{\mathbf{y}}[2d_{\lambda,\mathbf{y}}], \quad {}^\theta \mathcal{L}^\delta_{\mathbf{y}} = {}^\theta \mathcal{L}_{\mathbf{y}}[d_{\lambda,\mathbf{y}}].$$

For $\mathbf{i}, \mathbf{i}'$ in ${}^\theta I^\nu$ we consider the graded ${}^\theta \mathbf{S}_{\mathbf{V}}$-module

$${}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} = \operatorname{Ext}^*_{{}^\theta G_{\mathbf{V}}}({}^\theta \mathcal{L}^\vee_{\mathbf{i}}, {}^\theta \mathcal{L}^\vee_{\mathbf{i}'}).$$

The Yoneda composition is a homogeneous ${}^\theta \mathbf{S}_{\mathbf{V}}$-bilinear map of degree zero

$${}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} \times {}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}',\mathbf{i}''} \to {}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}''}, \quad \mathbf{i}, \mathbf{i}', \mathbf{i}'' \in {}^\theta I^\nu.$$

It equips the $\mathbf{k}$-vector space

$${}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}} = \bigoplus_{\mathbf{i},\mathbf{i}'\in {}^\theta I^\nu} {}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}$$

with the structure of a unital associative graded ${}^\theta \mathbf{S}_{\mathbf{V}}$-algebra. For $\mathbf{i} \in {}^\theta I^\nu$ we have the graded ${}^\theta \mathbf{S}_{\mathbf{V}}$-modules

$${}^\theta \mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} = \operatorname{Ext}^*_{{}^\theta G_{\mathbf{V}}}({}^\theta \mathcal{L}^\vee_{\mathbf{V},\mathbf{i}}, \mathcal{D}), \quad {}^\theta \mathbf{F}_{\mathbf{\Lambda},\mathbf{V}} = \bigoplus_{\mathbf{i}\in {}^\theta I^\nu} {}^\theta \mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}.$$

For each $\mathbf{i}, \mathbf{i}'$ in ${}^\theta I^\nu$ the Yoneda product gives a graded ${}^\theta \mathbf{S}_{\mathbf{V}}$-bilinear map ${}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} \times {}^\theta \mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'} \to {}^\theta \mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$. This yields a left graded representation of ${}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$ on ${}^\theta \mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$. Our first goal is to compute the graded algebra ${}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$ and the graded representation ${}^\theta \mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$. For $\mathbf{i} \in {}^\theta I^\nu$ let $1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ be the identity of ${}^\theta \mathcal{L}_{\mathbf{i}}$, regarded as an element of ${}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}}$. The elements $1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ form a complete set of orthogonal idempotents of ${}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$ such that

$${}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} = 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}\, {}^\theta \mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}\, 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}, \quad {}^\theta \mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} = 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}\, {}^\theta \mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}.$$

2.7. Remark. Fix a pair $\mathbf{y}$ in ${}^\theta Y^\nu$. Let $\mathbf{i}$ be the sequence of ${}^\theta I^\nu$ obtained by expanding $\mathbf{y}$. We have an isomorphism of complexes in the derived category

$$ {}^\theta\mathcal{L}^\delta_{\mathbf{i}} = \bigoplus_{w\in\mathfrak{S}_{\mathbf{b}}} {}^\theta\mathcal{L}^\delta_{\mathbf{y}}[\ell_{\mathbf{b}} - 2\ell(w)]. $$

Here $\mathbf{b} = (b_1, \dots, b_m)$ is a sequence such that the multiplicity of $\mathbf{y}$ is

$$ \theta(\mathbf{b})\mathbf{b} := (b_m, \dots, b_2, b_1, b_1, b_2, \dots b_m). $$

We'll abbreviate ${}^\theta\mathcal{L}^\delta_{\mathbf{i}} = \langle\mathbf{b}\rangle!\,{}^\theta\mathcal{L}^\delta_{\mathbf{y}}$.

2.8. Shift of the grading. Let λ, ν, $\mathbf{\Lambda}$, $\mathbf{V}$ be as above. We define a new grading on ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$ and ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ by

$$ {}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} = \mathrm{Ext}^*_{{}^\theta G_{\mathbf{V}}}({}^\theta\mathcal{L}^\delta_{\mathbf{i}}, {}^\theta\mathcal{L}^\delta_{\mathbf{i}'}) = {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}[d_{\lambda,\mathbf{i}} - d_{\lambda,\mathbf{i}'}], $$

$$ {}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}} = \bigoplus_{\mathbf{i},\mathbf{i}'\in{}^\theta I^\nu} {}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}, $$

$$ {}^\theta\mathcal{L}_{\mathbf{V}} = \bigoplus_{\mathbf{i}\in{}^\theta I^\nu} {}^\theta\mathcal{L}_{\mathbf{i}}, \quad {}^\theta\mathcal{L}^\delta_{\mathbf{V}} = \bigoplus_{\mathbf{i}\in{}^\theta I^\nu} {}^\theta\mathcal{L}^\delta_{\mathbf{i}}. $$

We set also ${}^\theta I^0 = \{\emptyset\}$, ${}^\theta\mathcal{L}^\delta_\emptyset = \mathbf{k}$, and ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\{0\}} = \mathbf{k}$ as a graded $\mathbf{k}$-algebra. Here $\mathbf{k}$ is regarded as the constant sheaf over $\{0\}$.

3. The convolution algebra

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector $\nu \neq 0$ in ${}^\theta\mathbb{N}I$ and a dimension vector λ in $\mathbb{N}I$. Fix an object $(\mathbf{V}, \varpi)$ in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ and an object $\mathbf{\Lambda}$ in $\boldsymbol{\mathcal{V}}_\lambda$. For each sequences $\mathbf{i}$, $\mathbf{i}'$ in ${}^\theta I^\nu$ we set

$$ {}^\theta Z_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} = {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} \times_{{}^\theta E_{\mathbf{\Lambda},\mathbf{V}}} {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}, \quad {}^\theta Z_{\mathbf{\Lambda},\mathbf{V}} = \coprod_{\mathbf{i},\mathbf{i}'\in{}^\theta I^\nu} {}^\theta Z_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}. $$

the reduced fiber product relative to the maps ${}^\theta\pi_{\mathbf{\Lambda},\mathbf{i}}$, ${}^\theta\pi_{\mathbf{\Lambda},\mathbf{i}'}$. Next we set

$$ {}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}} = \bigoplus_{\mathbf{i},\mathbf{i}'\in{}^\theta I^\nu} {}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}, \quad {}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}} = \bigoplus_{\mathbf{i}\in{}^\theta I^\nu} {}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, $$

where

$$ {}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} = H^{{}^\theta G_{\mathbf{V}}}_*({}^\theta Z_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}, \mathbf{k}), \quad {}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} = H^{{}^\theta G_{\mathbf{V}}}_*({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, \mathbf{k}). $$

We have

$$ {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} = \mathrm{Ext}^*_{{}^\theta G_{\mathbf{V}}}(\mathbf{k}, {}^\theta\mathcal{L}_{\mathbf{i}}) = H^*_{{}^\theta G_{\mathbf{V}}}({}^\theta E_{\mathbf{V}}, {}^\theta\mathcal{L}_{\mathbf{i}}) = H^*_{{}^\theta G_{\mathbf{V}}}({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, \mathbf{k}). $$

We have also

$$H^*_{{}^\theta G_{\mathbf{V}}}({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},\mathbf{k}) = H^*_{{}^\theta G_{\mathbf{V}}}({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},\mathcal{D})[-2d_{\lambda,\mathbf{i}}] = {}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}[-2d_{\lambda,\mathbf{i}}]. \tag{3.1}$$

This yields a graded ${}^\theta\mathbf{S}_{\mathbf{V}}$-module isomorphism

$${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} = {}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}[-2d_{\lambda,\mathbf{i}}]. \tag{3.2}$$

We equip the ${}^\theta\mathbf{S}_{\mathbf{V}}$-module ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$ with the convolution product relative to the closed embedding of ${}^\theta Z_{\mathbf{\Lambda},\mathbf{V}}$ into ${}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}\times{}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}$. See [CG, sec. 8.6] for details. We obtain an associative graded ${}^\theta\mathbf{S}_{\mathbf{V}}$-algebra ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$ with 1 which acts on the graded ${}^\theta\mathbf{S}_{\mathbf{V}}$-module ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$. The unit is the fundamental class of the closed subvariety ${}^\theta Z^e_{\mathbf{\Lambda},\mathbf{V}}$ of ${}^\theta Z_{\mathbf{\Lambda},\mathbf{V}}$. See Section 4.6 below for the notation.

3.1. Proposition. *(a) The left ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$-module ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$ is faithful.*

(b) There is a canonical ${}^\theta\mathbf{S}_{\mathbf{V}}$-algebra isomorphism ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}} = {}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$ such that (3.2) *identifies the ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$-action on ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ and the ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$-action on ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$.*

Proof : This is standard material, see e.g., [VV]. Let us give one proof of (a). It is a consequence of the following general fact. Let G be a linear algebraic group over $\mathbb{C}$ and let M be a smooth quasi-projective G-variety over $\mathbb{C}$. Let $T\subset G$ be a maximal torus. Let $\mathbf{Q}$ be the fraction field of $\mathbf{S}=\mathbf{S}_T$. Let $Z\subset M\times M$ be a closed G-stable subset (for the diagonal action on $M\times M$) such that $p_{1,3}$ restricts to a proper map

$$p_{1,2}^{-1}(Z)\cap p_{2,3}^{-1}(Z)\to Z,$$

where $p_{i,j}: M\times M\times M\to M\times M$ is the projection along the factor not named. The convolution product equips $H^G_*(Z,\mathbf{k})$ with a $\mathbf{S}_G$-algebra structure and $H^G_*(M,\mathbf{k})$ with a $H^G_*(Z,\mathbf{k})$-module structure, see e.g., [CG]. Assume now that the T-spaces M, Z are *equivariantly formal*, see e.g., [GKM, Sec. 1.2], and assume that we have the following equality of T-fixed points subsets

$$Z^T = M^T\times M^T. \tag{3.3}$$

Consider the following commutative diagram of algebra homomorphisms

$$\begin{tikzcd}
H^T_*(Z,\mathbf{k})\otimes_{\mathbf{S}}\mathbf{Q} \arrow[r, "c"] & \mathrm{End}_{\mathbf{S}}(H^T_*(M,\mathbf{k}))\otimes_{\mathbf{S}}\mathbf{Q} \\
H^T_*(Z,\mathbf{k}) \arrow[r] \arrow[u, "b"] & \mathrm{End}_{\mathbf{S}}(H^T_*(M,\mathbf{k})) \arrow[u] \\
H^G_*(Z,\mathbf{k}) \arrow[r] \arrow[u, "a"] & \mathrm{End}_{\mathbf{S}_G}(H^G_*(M,\mathbf{k})). \arrow[u]
\end{tikzcd}$$

The map c is invertible by (3.3) and the localization theorem in equivariant homology. The map b is injective because Z is equivariantly formal. The map a is injective, compare Section 4.10 below. Thus the lower map is injective, i.e., the $H^G_*(Z,\mathbf{k})$-module $H^G_*(M,\mathbf{k})$ is faithful.

□

4. The polynomial representation of the graded algebra ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector $\nu \neq 0$ in ${}^\theta\mathbb{N}I$ and a dimension vector λ in $\mathbb{N}I$. Set $|\nu| = 2m$. Fix an object $(\mathbf{V}, \varpi)$ in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ and an object $\mathbf{\Lambda}$ in $\boldsymbol{\mathcal{V}}_\lambda$. The main result of this section is Theorem 4.17 which yields an explicit faithful representation of the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$.

4.1. Notations. Let $G = O(\mathbf{V}, \varpi)$ be the orthogonal group, and $F = F(\mathbf{V}, \varpi)$ be the isotropic flag manifold. We can regard F as the (non connected) flag manifold of the (non connected) group G. Next, the group ${}^\theta G_\mathbf{V}$ is canonically identified with a Lévi subgroup of G, i.e., with the subgroup of elements which preserve the decomposition $\mathbf{V} = \bigoplus_i \mathbf{V}_i$. Then ${}^\theta F_\mathbf{V}$ is canonically identified with the closed subvariety of F consisting of all flags which are fixed under the action of the center of ${}^\theta G_\mathbf{V}$. Fix once for all a maximal torus T of ${}^\theta G_\mathbf{V}$. Let $W_\mathbf{V}$ and W be the Weyl groups of the pairs $({}^\theta G_\mathbf{V}, T)$ and (G, T). The canonical inclusion ${}^\theta G_\mathbf{V} \subset G$ yields a canonical inclusion $W_\mathbf{V} \subset W$.

4.2. The root systems. Fix once for all a T-fixed flag $\phi_\mathbf{V}$ in ${}^\theta F_\mathbf{V}$. We fix once for all one-dimensional T-submodules $\mathbf{D}_{1-m}, \dots, \mathbf{D}_{m-1}, \mathbf{D}_m$ of $\mathbf{V}$ such that

$$\phi_\mathbf{V} = (\mathbf{V}^l), \quad \mathbf{V}^l = \mathbf{D}_{l+1} \oplus \cdots \oplus \mathbf{D}_{m-1} \oplus \mathbf{D}_m.$$

Let $\chi_l \in \mathfrak{t}^*$ be the weight of $\mathbf{D}_l$. Note that $\mathbf{D}_l \simeq \mathbf{V}^{l-1}/\mathbf{V}^l$ and that the bilinear form ϖ yields a non-degenerate pairing $(\mathbf{V}^{l-1}/\mathbf{V}^l) \times (\mathbf{V}^{-l}/\mathbf{V}^{1-l}) \to \mathbb{C}$, because $(\mathbf{V}^l)^\perp = \mathbf{V}^{-l}$. Thus we have

$$\chi_{1-l} = -\chi_l.$$

Let B be the stabilizer of the flag $\phi_\mathbf{V}$ in G. Let Δ be the set of roots of (G, T) and let Δ^+ be the subset of positive roots relative to the Borel subgroup B. We abbreviate $\Delta^- = -\Delta^+$. Let Π be the set of simple roots in Δ^+. We have

$$\Delta^+ = \{\chi_k \pm \chi_l;\ 1 \leqslant l < k \leqslant m\},$$

$$\Pi = \{\chi_{l+1} - \chi_l,\ \chi_2 + \chi_1;\ l = 1, 2, \dots, m-1\}.$$

Let $\leqslant$ and ℓ denote the Bruhat order and the length function on W. Note that W is an extended Weyl group of type D_m. In particular we have

$$\ell(w) = 0 \iff w = e, \varepsilon_1,$$

where $\varepsilon_1$is as below, and the set S of simple reflections is given by

$$S = \{s_0, s_1, \dots, s_{m-1}\},$$

with s_k, $k = 0, 1, \dots, m-1$ the reflection with respect to

$$\alpha_0 = \chi_2 + \chi_1, \quad \alpha_1 = \chi_2 - \chi_1, \quad \dots \quad \alpha_{m-1} = \chi_m - \chi_{m-1}.$$

Note that $u(\Delta^+) = \Delta^+$ if $\ell(u) = 0$. Next, let ${}^\theta\Delta_\mathbf{V} \subset \Delta$ be the set of roots of $({}^\theta G_\mathbf{V}, T)$. Note that ${}^\theta G_\mathbf{V}$ is a product of general linear groups (this is due to the fact that θ has no fixed points). Indeed, we can (and we will) assume that

$${}^\theta\Delta_\mathbf{V} \subset \{\chi_l - \chi_k;\ l \neq k,\ l, k = 1, 2, \dots, m\}.$$

More precisely, given a subset $J \subset I$ such that $I = J \sqcup \theta(J)$ it is enough to choose the flag $\phi_{\mathbf{V}}$ such that $\mathbf{V}^0 = \bigoplus_{j\in J} \mathbf{V}_j$. Finally, let ${}^\theta\Delta^+_{\mathbf{V}}$ be the subset of positive roots relative to the Borel subgroup ${}^\theta B_{\mathbf{V}} = B \cap {}^\theta G_{\mathbf{V}}$. We have

$${}^\theta\Delta^+_{\mathbf{V}} = \Delta^+ \cap {}^\theta\Delta_{\mathbf{V}}.$$

4.3. The wreath product. Let $\mathfrak{S}_m$ be the symmetric group, and $\mathbb{Z}_2 = \{-1, 1\}$. Consider the wreath product $W_m = \mathfrak{S}_m \wr \mathbb{Z}_2$. For $l = 1, 2, \dots m$ let $\varepsilon_l \in (\mathbb{Z}_2)^m$ be -1 placed at the l-th position. We'll regard ε_l as in element of W_m in the obvious way. There is a unique action of W_m on the set $\{1-m, \dots, m-1, m\}$ such that $\mathfrak{S}_m$ permutes $1, 2, \dots m$ and such that ε_l fixes k if $k \neq l, 1-l$ and switches l and $1-l$. The group W_m acts also on ${}^\theta I^\nu$. Indeed, view a sequence $\mathbf{i}$ as the map

$$\{1-m, \dots, m-1, m\} \to I, \quad l \mapsto i_l.$$

Then we set $w(\mathbf{i}) = \mathbf{i} \circ w^{-1}$ for $w \in W_m$.

4.4. The W-action on the set of T-fixed flags. The sets F^T and $({}^\theta F_{\mathbf{V}})^T$ consisting of the flags which are fixed by the T-action are equal. The group W acts freely transitively on both. We'll write e for the unit in W. Put

$$\phi_{\mathbf{V},w} = w(\phi_{\mathbf{V}}), \quad \forall w \in W.$$

Thus we have $F^T = \{\phi_{\mathbf{V},w}; w \in W\}$. There is a unique group isomorphism $W = W_m$ such that

$$\phi_{\mathbf{V},w} = (\mathbf{V}^l_w), \quad \mathbf{V}^l_w = \mathbf{D}_{w(l+1)} \oplus \cdots \oplus \mathbf{D}_{w(m-1)} \oplus \mathbf{D}_{w(m)}.$$

We'll use this identification whenever it is convenient without recalling it explicitly. We set also

$$w(\chi_l) = \chi_{w(l)}, \quad \forall w, l.$$

Let ${}^\theta B_{\mathbf{V},w}$ be the stabilizer of the flag $\phi_{\mathbf{V},w}$ under the ${}^\theta G_{\mathbf{V}}$-action. It is the Borel subgroup of ${}^\theta G_{\mathbf{V}}$ containing T associated with the set of positive roots

$$w(\Delta^+) \cap {}^\theta\Delta_{\mathbf{V}}.$$

Let ${}^\theta N_{\mathbf{V},w}$ be the unipotent radical of ${}^\theta B_{\mathbf{V},w}$. Finally, let $\mathbf{i}_w$ be the unique sequence in ${}^\theta I^\nu$ such that $\phi_{\mathbf{V},w}$ lies in ${}^\theta F_{\mathbf{V},\mathbf{i}_w}$. Write

$$\mathbf{i}_e = (i_{1-m}, \dots, i_{m-1}, i_m). \tag{4.1}$$

Since $\phi_{\mathbf{V}}$ is a flag of type $\mathbf{i}_e$, we have

$$\mathbf{D}_l \subset \mathbf{V}_{i_l}, \quad w^{-1}(\mathbf{i}_e) = \mathbf{i}_w = (i_{w(1-m)}, \dots, i_{w(m-1)}, i_{w(m)}).$$

Let W_ν be the image of the group $W_{\mathbf{V}}$ by the isomorphism $W \to W_m$. It is the parabolic subgroup given by

$$W_\nu = \{w \in W_m; w(\mathbf{i}_e) = \mathbf{i}_e\}.$$

Note that the choices made in Section 4.2 imply that

$$W_\nu \subset \mathfrak{S}_m. \tag{4.2}$$

There is a bijection

$$W_\nu \setminus W_m \to {}^\theta I^\nu, \quad W_\nu w \mapsto \mathbf{i}_w.$$

For each $\mathbf{i}$ in ${}^\theta I^\nu$ we have

$$({}^\theta \widetilde{F}_{\mathbf{V},\mathbf{i}})^T \simeq ({}^\theta F_{\mathbf{V},\mathbf{i}})^T = \{\phi_{\mathbf{V},w}; w \in W_\mathbf{i}\}, \quad W_\mathbf{i} = \{w \in W; \mathbf{i}_w = \mathbf{i}\}.$$

We'll abbreviate

$${}^\theta F_{\mathbf{V},w} = {}^\theta F_{\mathbf{V},\mathbf{i}_w}, \quad W_w = W_{\mathbf{i}_w}, \quad {}^\theta \pi_{\mathbf{\Lambda},w} = {}^\theta \pi_{\mathbf{\Lambda},\mathbf{i}_w}.$$

We'll also omit the symbol w if $w = e$. For instance we write ${}^\theta B_\mathbf{V} = {}^\theta B_{\mathbf{V},e}$ and ${}^\theta N_\mathbf{V} = {}^\theta N_{\mathbf{V},e}$. Note that $W_w = W_\mathbf{V}\, w$ and that we have an isomorphism of ${}^\theta G_\mathbf{V}$-varieties

$${}^\theta G_\mathbf{V} / {}^\theta B_{\mathbf{V},w} \to {}^\theta F_{\mathbf{V},w}, \quad g \mapsto g\phi_{\mathbf{V},w}.$$

4.5. The stratification of ${}^\theta F_\mathbf{V} \times {}^\theta F_\mathbf{V}$. The group G acts diagonally on $F \times F$. The action of the subgroup ${}^\theta G_\mathbf{V}$ preserves the subset ${}^\theta F_\mathbf{V} \times {}^\theta F_\mathbf{V}$. For $w \in W$ let ${}^\theta O^w_\mathbf{V}$ be the set of all pairs of flags in ${}^\theta F_\mathbf{V} \times {}^\theta F_\mathbf{V}$ which are in relative position w. More precisely, we write

$${}^\theta O^w_\mathbf{V} = ({}^\theta F_\mathbf{V} \times {}^\theta F_\mathbf{V}) \cap (G\phi_{\mathbf{V},e,w}), \quad \phi_{\mathbf{V},x,y} = (\phi_{\mathbf{V},x}, \phi_{\mathbf{V},y}), \quad \forall x, y \in W.$$

Let ${}^\theta \bar{O}^w_\mathbf{V}$ be the Zariski closure of ${}^\theta O^w_\mathbf{V}$. For any w, x, y in W we write also

$${}^\theta O^w_{\mathbf{V},x,y} = {}^\theta O^w_\mathbf{V} \cap ({}^\theta F_{\mathbf{V},x} \times {}^\theta F_{\mathbf{V},y}), \quad {}^\theta \bar{O}^w_{\mathbf{V},x,y} = {}^\theta \bar{O}^w_\mathbf{V} \cap ({}^\theta F_{\mathbf{V},x} \times {}^\theta F_{\mathbf{V},y}).$$

We define ${}^\theta P_{\mathbf{V},w,ws}$, $s \in S$, as the smallest parabolic subgroup of ${}^\theta G_\mathbf{V}$ containing ${}^\theta B_{\mathbf{V},w}$ and ${}^\theta B_{\mathbf{V},ws}$.

4.6. Lemma. *Let $w, x, y, s, u \in W$.*

(a) The set of T-fixed elements in ${}^\theta O^x_\mathbf{V}$ is $\{\phi_{\mathbf{V},w,wx};\ w \in W\}$.

(b) Assume that $\ell(u) = 0$. We have ${}^\theta \bar{O}^u_\mathbf{V} = {}^\theta O^u_\mathbf{V}$. It is a smooth ${}^\theta G_\mathbf{V}$-variety isomorphic to ${}^\theta F_\mathbf{V}$. We have ${}^\theta O^u_{\mathbf{V},x,y} = \emptyset$ unless $y = xu$.

(c) Assume that $\ell(s) = 1$. Set $s = s'u$ with $s' \in S$ and $\ell(u) = 0$. We have ${}^\theta \bar{O}^s_\mathbf{V} = {}^\theta O^s_\mathbf{V} \cup {}^\theta O^u_\mathbf{V}$. It is a smooth variety. We have ${}^\theta \bar{O}^s_{\mathbf{V},x,y} = \emptyset$ if $y \neq xs, xu$.

- *If $xs \notin W_{xu}$ then*

$${}^\theta F_{\mathbf{V},xs} \neq {}^\theta F_{\mathbf{V},xu}, \quad {}^\theta B_{\mathbf{V},xs} = {}^\theta B_{\mathbf{V},x}, \quad {}^\theta O^u_{\mathbf{V},x,xs} = {}^\theta O^s_{\mathbf{V},x,xu} = \emptyset.$$

- *If $xs \in W_{xu}$ then*

$${}^\theta F_{\mathbf{V},xs} = {}^\theta F_{\mathbf{V},xu}, \quad {}^\theta B_{\mathbf{V},xs} \neq {}^\theta B_{\mathbf{V},x},$$

$${}^\theta G_\mathbf{V} \times_{{}^\theta B_{\mathbf{V},x}} ({}^\theta P_{\mathbf{V},x,xs} / {}^\theta B_{\mathbf{V},x}) \xrightarrow{\sim} {}^\theta \bar{O}^s_{\mathbf{V},x,xu} = {}^\theta \bar{O}^s_{\mathbf{V},x,xs}, \ (g,h) \mapsto (g\phi_{\mathbf{V},x}, gh\phi_{\mathbf{V},xu}).$$

Proof : The proof is standard and is left to the reader. Note that ${}^\theta B_{\mathbf{V},x} = {}^\theta B_{\mathbf{V},xu}$ and ${}^\theta B_{\mathbf{V},xs} = {}^\theta B_{\mathbf{V},xs'}$ because $\ell(u) = 0$. Note also that

$$ {}^\theta B_{\mathbf{V},x} = {}^\theta B_{\mathbf{V},xs} \iff x(\alpha) \notin {}^\theta\Delta_{\mathbf{V}} \iff xs' \in W_x \iff xs \in W_{xu}, $$

where α is the simple root associated with s'.

□

For a future use let us introduce the following notation. Let q be the obvious projection ${}^\theta Z_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta F_{\mathbf{V}} \times {}^\theta F_{\mathbf{V}}$, and, for each $x \in W$, let ${}^\theta Z^x_{\mathbf{\Lambda},\mathbf{V}}$ be the Zariski closure in ${}^\theta Z_{\mathbf{\Lambda},\mathbf{V}}$ of the locally closed subset $q^{-1}({}^\theta O^x_{\mathbf{V}})$.

4.7. Euler classes in S. Consider the graded $\mathbf{k}$-algebra $\mathbf{S} = \mathbf{S}_T$. The weights $\chi_1, \chi_2, \dots \chi_m$ are algebraically independent generators of $\mathbf{S}$ and they are homogeneous of degree 2. The reflection representation on $\mathfrak{t}$ yields a W-action on $\mathbf{S}$. Recall that we have

$$ w(\chi_l) = \chi_{w(l)}, \quad \forall l, w. $$

Now, let M be a finite dimensional representation of $\mathfrak{t}$ and fix a linear form $\lambda \in \mathfrak{t}^*$. Let $M[\lambda] \subset M$ be the weight subspace associated with λ. The *character* of M is the linear form $\mathrm{ch}(M) = \sum_\lambda \dim(M[\lambda])\,\lambda$. Let $\mathrm{eu}(M)$ be the determinant of M, viewed as an element of degree $2\mathrm{dim}(M)$ of $\mathbf{S}$. We'll call $\mathrm{eu}(M)$ the *Euler class of* M. If M is a finite dimensional representation of T let $\mathrm{eu}(M)$ be the Euler class of the differential of M, a module over $\mathfrak{t}$. Now, assume that X is a quasi-projective T-variety and that $x \in X^T$ is a smooth point of X. The cotangent space $T^*_x X$ at x is equipped with a natural representation of T. We'll abbreviate $\mathrm{eu}(X, x) = \mathrm{eu}(T^*_x X)$. We'll be particularly interested in the following elements

$$ \Lambda_w = \mathrm{eu}({}^\theta \widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}, \phi_{\mathbf{V},w}), \quad \Lambda^x_{w,w'} = \mathrm{eu}({}^\theta Z^x_{\mathbf{\Lambda},\mathbf{V}}, \phi_{\mathbf{V},w,w'})^{-1} $$

where $\ell(x) = 0, 1$. Note that Λ_w lies in $\mathbf{S}$ and has the degree $2d_{\lambda,w}$.

4.8. Description of the ${}^\theta G_{\mathbf{V}}$-varieties ${}^\theta \widetilde{F}_{\mathbf{\Lambda},\mathbf{V},w}$. Let ${}^\theta\mathfrak{g}_{\mathbf{V}}$, $\mathfrak{t}$, ${}^\theta\mathfrak{n}_{\mathbf{V},w}$, $w \in W$, be the Lie algebras of ${}^\theta G_{\mathbf{V}}$, T, ${}^\theta N_{\mathbf{V},w}$ respectively. Consider the flag

$$ \phi_{\mathbf{V},w} = (\mathbf{V} = \mathbf{V}^{-m}_w \supset \cdots \supset \mathbf{V}^{m-1}_w \supset \mathbf{V}^m_w = 0). $$

The ${}^\theta G_{\mathbf{V}}$-action on ${}^\theta E_{\mathbf{\Lambda},\mathbf{V}}$ yields a representation of ${}^\theta B_{\mathbf{V},w}$ on the space

$$ {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w} = \{(x, y) \in {}^\theta E_{\mathbf{\Lambda},\mathbf{V}};\, x(\mathbf{V}^l_w) \subset \mathbf{V}^l_w,\, y(\mathbf{\Lambda}) \subset \mathbf{V}^m_w\}. $$

There is an isomorphism of ${}^\theta G_{\mathbf{V}}$-varieties

$$ {}^\theta G_{\mathbf{V}} \times_{{}^\theta B_{\mathbf{V},w}} {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w} \to {}^\theta \widetilde{F}_{\mathbf{\Lambda},\mathbf{V},w}, \quad (g, x, y) \mapsto (g\phi_{\mathbf{V},w}, gx, gy). $$

Under this isomorphism the map ${}^\theta\pi_{\mathbf{\Lambda},w}$ is identified with the map

$$ {}^\theta G_{\mathbf{V}} \times_{{}^\theta B_{\mathbf{V},w}} {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w} \to {}^\theta E_{\mathbf{\Lambda},\mathbf{V}}, \quad (g, x, y) \mapsto (gx, gy). $$

4.9. Character formulas. In this section we gather some character formula for a later use. For $w, w' \in W$ we write

$$ {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w,w'} = {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w} \cap {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w'}, $$

$$ {}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w,w'} = {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w} / {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w,w'}, $$

$$ {}^\theta\mathfrak{n}_{\mathbf{V},w,w'} = {}^\theta\mathfrak{n}_{\mathbf{V},w} \cap {}^\theta\mathfrak{n}_{\mathbf{V},w'}, $$

$$ {}^\theta\mathfrak{m}_{\mathbf{V},w,w'} = {}^\theta\mathfrak{n}_{\mathbf{V},w} / {}^\theta\mathfrak{n}_{\mathbf{V},w,w'}. $$

We have the following T-module isomorphisms

$$ {}^\theta\mathfrak{n}_{\mathbf{V},w} = {}^\theta\mathfrak{n}_{\mathbf{V},w,w'} \oplus {}^\theta\mathfrak{m}_{\mathbf{V},w,w'}, $$

$$ {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w} = {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w,w'} \oplus {}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w,w'}, $$

$$ {}^\theta\mathfrak{m}_{\mathbf{V},w,w'} = ({}^\theta\mathfrak{m}_{\mathbf{V},w',w})^*. $$

Write ${}^\theta\mathfrak{e}_{\mathbf{V},w} = {}^\theta\mathfrak{e}_{\{0\},\mathbf{V},w}$. As T-modules we have

$$ \begin{aligned} {}^\theta\mathfrak{n}_{\mathbf{V},w} &= \bigoplus_\alpha \mathfrak{g}[\alpha], \quad \alpha \in w(\Delta^+) \cap {}^\theta\Delta_{\mathbf{V}}, \\ {}^\theta\mathfrak{e}_{\mathbf{V},w} &= \bigoplus_\alpha {}^\theta E_{\mathbf{V}}[\alpha], \quad \alpha \in w(\Delta^+). \end{aligned} \tag{4.3} $$

Recall that $\mathbf{V} = \bigoplus_l \mathbf{D}_l$ as I-graded T-modules, where $l = 1-m, \dots, m-1, m$. We'll use the notation in (4.1). Thus i_l, χ_l are the dimension vector and the character of $\mathbf{D}_l$. Note that

$$ h_{i_k,i_l} = h_{i_{1-l},i_{1-k}}, \quad \chi_l = -\chi_{1-l}, $$

$$ \nu_{\theta(i)} = \nu_i, \quad \nu = \sum_i \nu_i \, i = \sum_{l=1}^m (i_l + i_{1-l}). $$

Set $H^0 = \{h \in H;\, h' = \theta(h'')\}$, $H^1 = H \setminus H^0$, and $\lambda = \sum_i \lambda_i \, i$. Note that $H^0 = \{h \in H;\, h = \theta(h)\}$. Decomposing a tuple $x \in {}^\theta E_{\mathbf{V}}$ as the sum of $(x_h)_{h\in H^1}$ and $(x_h)_{h\in H^0}$ we get the following formula

$$ \dim({}^\theta E_{\mathbf{V}}) = \sum_{h\in H^1} \nu_{h'}\nu_{h''}/2 + \sum_{h\in H^0} \nu_{h'}(\nu_{h'} - 1)/2. $$

Next, the decomposition (4.3) yields the following formula

$$ \mathrm{ch}({}^\theta\mathfrak{e}_{\mathbf{V},w}) = \sum_{\chi_l - \chi_k \in w(\Delta^+)} h_{i_k,i_l}(\chi_l - \chi_k). $$

Here the sum runs over $\alpha \in w(\Delta^+)$, and for each α we choose one pair (l,k) such that $\alpha = \chi_l - \chi_k$. In a similar way we have also

$$ \mathrm{ch}({}^\theta E_{\mathbf{V}}) = \sum_{\chi_l - \chi_k \in \Delta} h_{i_k,i_l}(\chi_l - \chi_k), $$

$$ \mathrm{ch}(L_{\mathbf{\Lambda},\mathbf{V}}) = \sum_l \lambda_{i_l} \chi_l. $$

Here the first sum runs over Δ. Since $\mathbf{V}^0 = \bigoplus_{l\geqslant 1} \mathbf{D}_l$ we have also

$$\mathrm{ch}({}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w}) = \sum_{\chi_l-\chi_k} h_{i_k,i_l}(\chi_l - \chi_k) + \sum_l \lambda_{i_l}\chi_l, \tag{4.4}$$

where the first sum runs over all roots in $w(\Delta^+)$ and the second one over all l in $\{w(1), w(2), \dots, w(m)\}$. Note that (4.4) can be rewriten in the following way

$$\mathrm{ch}({}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w}) = \sum_{\chi_l-\chi_k\in\Delta^+} h_{i_{w(k)},i_{w(l)}} w(\chi_l - \chi_k) + \sum_{1\leqslant l\leqslant m} \lambda_{i_{w(l)}} w(\chi_l).$$

By (4.3) the Euler class $\mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},w})$ is the product of all roots in ${}^\theta\Delta_{\mathbf{V}} \cap w(\Delta^+)$. Therefore, for $s \in S$ the following formulas hold

- either $ws \notin W_w$ and we have
$$\mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},ws}) = \mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},w}),$$
$$\mathrm{eu}({}^\theta\mathfrak{m}_{\mathbf{V},w,ws}) = \mathrm{eu}({}^\theta\mathfrak{m}_{\mathbf{V},ws,w}) = 0,$$
- or $ws \in W_w$ and we have
$$\mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},ws}) = -\mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},w}),$$
$$\mathrm{eu}({}^\theta\mathfrak{m}_{\mathbf{V},w,ws}) = -\mathrm{eu}({}^\theta\mathfrak{m}_{\mathbf{V},ws,w}) = w(\alpha),$$
where α is the simple root associated with s.

Finally, let $s = s_l$ with $l = 0, 1, \dots, m-1$. Formula (4.4) yields the following.

- We have
$$\mathrm{eu}({}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w,w\varepsilon_1}) = w(\chi_1)^{\lambda_{i_{w(1)}}}.$$
- If $l \neq 0$ we have
$$\mathrm{eu}({}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w,ws_l}) = w(\alpha_l)^{h_{i_{w(l)},i_{w(l+1)}}}.$$
- We have
$$\mathrm{eu}({}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w,ws_0}) = w(\chi_1)^{\lambda_{i_{w(1)}}} w(\chi_2)^{\lambda_{i_{w(2)}}} w(\alpha_0)^{h_{i_{w(0)},i_{w(2)}}}.$$

4.10. Reduction to the torus. The restriction of functions from ${}^\theta\mathfrak{g}_{\mathbf{V}}$ to $\mathfrak{t}$ gives an isomorphism of graded $\mathbf{k}$-algebras

$${}^\theta\mathbf{S}_{\mathbf{V}} = \mathbf{k}[\chi_1, \chi_2, \dots, \chi_m]^{W_\nu}.$$

The group ${}^\theta G_{\mathbf{V}}$ is a product of several copies of the general linear group. Hence it is connected with a simply connected derived subgroup. It is a general fact that if X is a ${}^\theta G_{\mathbf{V}}$-variety then the $\mathbf{S}$-module $H_*^T(X, \mathbf{k})$ is equipped with a $\mathbf{S}$-skewlinear representation of the group $W_{\mathbf{V}}$ such that the forgetful map gives a ${}^\theta\mathbf{S}_{\mathbf{V}}$-module isomorphism

$$H_*^{{}^\theta G_{\mathbf{V}}}(X, \mathbf{k}) \to H_*^T(X, \mathbf{k})^{W_{\mathbf{V}}},$$

see e.g., [HS, thm. 2.10]. We'll call this action on $H_*^T(X, \mathbf{k})$ the canonical $W_{\mathbf{V}}$-action.

4.11. The W-action and the ${}^\theta\mathbf{S}_\mathbf{V}$-action on ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$. Fix a tuple $\mathbf{i}$ in ${}^\theta I^\nu$ and an integer $l = 1, 2, \dots, m$. We define $\mathcal{O}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l)$ to be the ${}^\theta G_\mathbf{V}$-equivariant line bundle over ${}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ whose fiber at the triple (x, y, ϕ) with

$$\phi = (\mathbf{V} = \mathbf{V}^{-m} \supset \mathbf{V}^{1-m} \supset \cdots \supset \mathbf{V}^m = 0)$$

is equal to $\mathbf{V}^{l-1}/\mathbf{V}^l$. Assigning to a formal variable $x_\mathbf{i}(l)$ of degree 2 the first equivariant Chern class of $\mathcal{O}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l)^{-1}$ we get a graded $\mathbf{k}$-algebra isomorphism

$$\mathbf{k}[x_\mathbf{i}(1), x_\mathbf{i}(2), \dots x_\mathbf{i}(m)] = H^*_{{}^\theta G_\mathbf{V}}({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, \mathbf{k}).$$

So (3.1), (3.2) yield canonical isomorphisms of graded $\mathbf{k}$-vector spaces

$$\mathbf{k}[x_\mathbf{i}(1), x_\mathbf{i}(2), \dots x_\mathbf{i}(m)] = {}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}[-2d_{\lambda,\mathbf{i}}] = {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}. \tag{4.5}$$

For a future use we set also

$$x_\mathbf{i}(l) = -x_\mathbf{i}(1-l), \quad l = 1-m, 2-m, \dots, 0.$$

For $w \in W_m$ we set

$$wf(x_\mathbf{i}(1), \dots, x_\mathbf{i}(m)) = f(x_{w(\mathbf{i})}(w(1)), \dots, x_{w(\mathbf{i})}(w(m))).$$

This yields a W_m-action on ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ such that $w({}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}) = {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},w(\mathbf{i})}$.

The multiplication of polynomials equip both ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ and ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ with an obvious structure of graded $\mathbf{k}$-algebras. For $w \in W_\mathbf{i}$ the pull-back by the inclusion $\{\phi_{\mathbf{V},w}\} \subset {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ yields a graded $\mathbf{k}$-algebra isomorphism

$$ {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} \to \mathbf{S}, \quad f(-x_\mathbf{i}(1), \dots, -x_\mathbf{i}(m)) \mapsto f(\chi_{w(1)}, \dots \chi_{w(m)}). \tag{4.6}$$

We'll abbreviate

$$w(f) = f(\chi_{w(1)}, \dots \chi_{w(m)}).$$

The isomorphism (4.6) is not canonical, because it depends on the choice of w.

Now, consider the canonical ${}^\theta\mathbf{S}_\mathbf{V}$-action on ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ coming from the ${}^\theta G_\mathbf{V}$-equivariant cohomology. It can be regarded as a ${}^\theta\mathbf{S}_\mathbf{V}$-action on $\bigoplus_\mathbf{i} \mathbf{k}[x_\mathbf{i}(1), x_\mathbf{i}(2), \dots x_\mathbf{i}(m)]$ which is described in the following way. The composition of the obvious projection ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ with the map (4.6) identifies the graded $\mathbf{k}$-algebra of the W_m-invariant polynomials in the $x_\mathbf{i}(l)$'s, with

$${}^\theta\mathbf{S}_\mathbf{V} = \mathbf{S}^{W_\nu} = \mathbf{k}[\chi_1, \chi_2, \dots, \chi_m]^{W_\nu}.$$

This isomorphism does not depend on the choice of $\mathbf{i}, w$. The ${}^\theta\mathbf{S}_\mathbf{V}$-action on ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ is the composition of this isomorphism and of the multiplication by W_m-invariant polynomials.

4.12. Localization and the convolution product. Let $\mathbf{Q}$ be the fraction field of $\mathbf{S}$. Write

$$
\begin{aligned}
{}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}} &= H^T_*({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}},\mathbf{k}),\\
{}^\theta\mathcal{F}''_{\mathbf{\Lambda},\mathbf{V}} &= {}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}}\otimes_{\mathbf{S}}\mathbf{Q},\\
{}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}} &= H^T_*({}^\theta Z_{\mathbf{\Lambda},\mathbf{V}},\mathbf{k}),\\
{}^\theta\mathcal{Z}''_{\mathbf{\Lambda},\mathbf{V}} &= {}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}\otimes_{\mathbf{S}}\mathbf{Q}.
\end{aligned}
$$

Let ψ_w be the fundamental class of the singleton $\{\phi_{\mathbf{V},w}\}$ in ${}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}}$, and let $\psi_{w,w'}$ be the fundamental class of $\{\phi_{\mathbf{V},w,w'}\}$ in ${}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}$. Let ψ_w, $\psi_{w,w'}$ denote also the corresponding elements in the $\mathbf{Q}$-vector spaces ${}^\theta\mathcal{F}''_{\mathbf{\Lambda},\mathbf{V}}$, ${}^\theta\mathcal{Z}''_{\mathbf{\Lambda},\mathbf{V}}$. Now, we consider the convolution products

$$
{}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}\times{}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}\to{}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}},\quad {}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}\times{}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}}\to{}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}}
$$

relative to the inclusion of ${}^\theta Z_{\mathbf{\Lambda},\mathbf{V}}$ in the smooth scheme ${}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}\times{}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}$. Both may be denoted by the symbol $\star$. We'll use the notation in (4.1).

4.13. Proposition. *(a) The $\mathbf{S}$-modules ${}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}}$ and ${}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}$ are free. The canonical $W_{\mathbf{V}}$-action on the T-equivariant homology spaces ${}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}}$ and ${}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}$ is given by $w(\psi_x)=\psi_{wx}$ and $w(\psi_{x,y})=\psi_{wx,wy}$. The inclusions ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}\subset{}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}$ and ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}\subset{}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}}$ commute with the convolution products.*

(b) The elements ψ_w, $\psi_{w,w'}$ yield $\mathbf{Q}$-bases of ${}^\theta\mathcal{F}''_{\mathbf{\Lambda},\mathbf{V}}$, ${}^\theta\mathcal{Z}''_{\mathbf{\Lambda},\mathbf{V}}$ respectively. For each $\mathbf{i}$ the map (4.5) yields an inclusion of $\mathbf{k}[x_{\mathbf{i}}(1),\dots,x_{\mathbf{i}}(m)]$ into ${}^\theta\mathcal{F}''_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ such that

$$
f(-x_{\mathbf{i}}(1),\dots,-x_{\mathbf{i}}(m))\mapsto\sum_{w\in W_{\mathbf{i}}}w(f)\Lambda_w^{-1}\psi_w.
$$

(c) We have $\psi_{w',w}\star\psi_w=\Lambda_w\,\psi_{w'}$ and $\psi_{w'',w'}\star\psi_{w',w}=\Lambda_{w'}\,\psi_{w'',w}$.

(d) If $\ell(s)=0,1$ then $[{}^\theta Z^s_{\mathbf{\Lambda},\mathbf{V}}]=\sum_{w,w'}\Lambda^s_{w,w'}\,\psi_{w,w'}$ in ${}^\theta\mathcal{Z}''_{\mathbf{\Lambda},\mathbf{V}}$.

(e) We have $\Lambda_w=\mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w}\oplus{}^\theta\mathfrak{n}_{\mathbf{V},w})$.

(f) If $\ell(u)=0$ then $\Lambda^u_{w,w'}=0$ if $w'\neq wu$, and

$$
\Lambda^e_{w,w}=\Lambda_w^{-1},\quad \Lambda^{\varepsilon_1}_{w,w\varepsilon_1}=\big(\chi_{w(0)}\big)^{\lambda_{i_{w(1)}}}\Lambda_w^{-1}=\big(\chi_{w(1)}\big)^{\lambda_{i_{w(0)}}}\Lambda_{w\varepsilon_1}^{-1}.
$$

(g) If $l=0,1,\dots,m-1$ then

- *either $ws_l\notin W_w$ and*

$$
\Lambda^{s_l}_{w,ws_l}=\Lambda^{s_l}_{w,w}=\mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w,ws_l}\oplus{}^\theta\mathfrak{n}_{\mathbf{V},w})^{-1},
$$

- *or $ws_l\in W_w$ and*

$$
\begin{aligned}
\Lambda^{s_l}_{w,w}&=\mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w,ws_l}\oplus{}^\theta\mathfrak{n}_{\mathbf{V},w}\oplus{}^\theta\mathfrak{m}_{\mathbf{V},w,ws_l})^{-1},\\
\Lambda^{s_l}_{w,ws_l}&=\mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w,ws_l}\oplus{}^\theta\mathfrak{n}_{\mathbf{V},w}\oplus{}^\theta\mathfrak{m}_{\mathbf{V},ws_l,w})^{-1}.
\end{aligned}
$$

Proof : Parts (a) to (d) are left to the reader. The fiber at $\phi_{\mathbf{V},w}$ of the vector bundle

$$p : {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta F_{\mathbf{V}}$$

is isomorphic to ${}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w}$ as a T-module. Thus the cotangent space to ${}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}$ at the point $\phi_{\mathbf{V},w}$ is isomorphic to ${}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w}$ as a T-module. This yields (e). Next, observe that the variety ${}^\theta Z^s_{\mathbf{\Lambda},\mathbf{V}}$ is smooth if $\ell(s) \leqslant 1$. First, assume that $\ell(u) = 0$. The fiber at $\phi_{\mathbf{V},w,w'}$ of the vector bundle

$$q : {}^\theta Z^u_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta F_{\mathbf{V}} \times {}^\theta F_{\mathbf{V}}$$

is isomorphic to ${}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w,wu}$ as a T-module if $w' = wu$ and it is zero else. Thus we have

$$\begin{aligned}\Lambda^u_{w,wu} &= \mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w,wu})\mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w})^{-1}\mathrm{eu}({}^\theta F_{\mathbf{V}}, \phi_{\mathbf{V},w})^{-1},\\ &= \mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w,wu})\Lambda_w^{-1}.\end{aligned}$$

Therefore, Section 4.9 yields

$$\Lambda^u_{w,wu} = \begin{cases} \Lambda_w^{-1} & \text{if } u = e, \\ (-\chi_{w(1)})^{\lambda_{i_{w(1)}}}\Lambda_w^{-1} & \text{if } u = \varepsilon_1. \end{cases}$$

Note that

$$\chi_{w(1-l)} = -\chi_{w(l)}, \quad \forall w, l.$$

This yields (f). Finally, let us concentrate on part (g). The fiber at $\phi_{\mathbf{V},w,w'}$ of the vector bundle

$$q : {}^\theta Z^{s_l}_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta F_{\mathbf{V}} \times {}^\theta F_{\mathbf{V}}$$

is isomorphic to ${}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w,ws_l}$ as a T-module if $w' = w, ws_l$ and it is zero else. Therefore, we have

$$\Lambda^{s_l}_{w,w'} = \begin{cases} \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w,ws_l})^{-1}\mathrm{eu}({}^\theta\bar{O}^{s_l}_{\mathbf{V}}, \phi_{\mathbf{V},w,w'})^{-1} & \text{if } w' = w, ws_l \\ 0 & \text{else.} \end{cases}$$

Next, by Lemma 4.6(c), if $ws_l \notin W_w$ the cotangent spaces to the variety ${}^\theta\bar{O}^{s_l}_{\mathbf{V}}$ at the points $\phi_{\mathbf{V},w,ws_l}$ and $\phi_{\mathbf{V},w,w}$ are given by

$$\begin{aligned} T^*_{w,ws_l}{}^\theta\bar{O}^{s_l}_{\mathbf{V}} &= T^*_{w,ws_l}{}^\theta O^{s_l}_{\mathbf{V}} = {}^\theta\mathfrak{n}_{\mathbf{V},w,ws_l} = {}^\theta\mathfrak{n}_{\mathbf{V},w},\\ T^*_{w,w}{}^\theta\bar{O}^{s_l}_{\mathbf{V}} &= T^*_{w,w}{}^\theta O^{e}_{\mathbf{V}} = {}^\theta\mathfrak{n}_{\mathbf{V},w,w} = {}^\theta\mathfrak{n}_{\mathbf{V},w}.\end{aligned}$$

Thus they are both isomorphic to ${}^\theta\mathfrak{n}_{\mathbf{V},w}$ as T-modules. Similarly, if $ws_l \in W_w$ the cotangent spaces to the variety ${}^\theta\bar{O}^{s_l}_{\mathbf{V}}$ at the points $\phi_{\mathbf{V},w,ws_l}$, $\phi_{\mathbf{V},w,w}$ are given by

$$\begin{aligned} T^*_{w,ws_l}{}^\theta\bar{O}^{s_l}_{\mathbf{V}} &= {}^\theta\mathfrak{n}_{\mathbf{V},w} \oplus {}^\theta\mathfrak{m}_{\mathbf{V},ws_l,w},\\ T^*_{w,w}{}^\theta\bar{O}^{s_l}_{\mathbf{V}} &= {}^\theta\mathfrak{n}_{\mathbf{V},w} \oplus {}^\theta\mathfrak{m}_{\mathbf{V},w,ws_l},\end{aligned}$$

because $\mathrm{Lie}({}^\theta P_{\mathbf{V},w,ws_l})/\mathrm{Lie}({}^\theta B_{\mathbf{V},w})$ is dual to ${}^\theta\mathfrak{m}_{\mathbf{V},w,ws_l} = {}^\theta\mathfrak{n}_{\mathbf{V},w}/{}^\theta\mathfrak{n}_{\mathbf{V},w,ws_l}$ as a T-module.

□

4.14. Description of the ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$-action on ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$. Using the computations in the previous proposition we can now describe explicitly the representation of ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$ in ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$. For $k=0,1,\dots,m-1$ let $\sigma_{\mathbf{\Lambda},\mathbf{V}}(k)$ be the fundamental class of ${}^\theta Z^{s_k}_{\mathbf{\Lambda},\mathbf{V}}$ in ${}^\theta\mathcal{Z}^{\leqslant s_k}_{\mathbf{\Lambda},\mathbf{V}}$. Next, let $\pi_{\mathbf{\Lambda},\mathbf{V}}(1)$ be the fundamental class of ${}^\theta Z^{\varepsilon_1}_{\mathbf{\Lambda},\mathbf{V}}$ in ${}^\theta\mathcal{Z}^{\varepsilon_1}_{\mathbf{\Lambda},\mathbf{V}}$. Finally, for $l=1,2,\dots,m$ the pull-back of the first equivariant Chern class of the line bundle $\bigoplus_{\mathbf{i}}\mathcal{O}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l)^{-1}$ by the obvious map

$$ {}^\theta Z^e_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}} $$

belongs to $H^*_{{}^\theta G_{\mathbf{V}}}({}^\theta Z^e_{\mathbf{\Lambda},\mathbf{V}},\mathbf{k})$. So it yields an element $\varkappa_{\mathbf{\Lambda},\mathbf{V}}(l)$ in ${}^\theta\mathcal{Z}^e_{\mathbf{\Lambda},\mathbf{V}}$. Now, recall that ${}^\theta\mathcal{Z}^{\leqslant w}_{\mathbf{\Lambda},\mathbf{V}}$ embeds into ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$. Thus the classes $\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}',\mathbf{i}}(k)$, $\pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}',\mathbf{i}}(1)$ and $\varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}',\mathbf{i}}(l)$ can all be regarded as elements of ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$. We write

$$ \begin{aligned} \sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}',\mathbf{i}}(k) &= 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}\star\sigma_{\mathbf{\Lambda},\mathbf{V}}(k)\star 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},\\ \pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}',\mathbf{i}}(1) &= 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}\star\pi_{\mathbf{\Lambda},\mathbf{V}}(1)\star 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},\\ \varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}',\mathbf{i}}(l) &= 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}\star\varkappa_{\mathbf{\Lambda},\mathbf{V}}(l)\star 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}. \end{aligned} $$

For a sequence $\mathbf{i}=(i_{1-m},\dots,i_{m-1},i_m)$ and integers $l=1-m,\dots,m-1,m$ and $k=1,\dots,m-1,m$, we set

$$ \lambda_{\mathbf{i}}(l)=\lambda_{i_l},\quad h_{\mathbf{i}}(k)=\begin{cases} -1 & \text{if } s_k\mathbf{i}=\mathbf{i},\\ h_{i_k,i_{k+1}} & \text{if } s_k\mathbf{i}\neq\mathbf{i},\ k\neq 0,\\ h_{i_0,i_2} & \text{if } s_0\mathbf{i}\neq\mathbf{i},\ k=0. \end{cases} $$

Finally, recall that ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$ acts on ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}=\bigoplus_{\mathbf{i}}{}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ and that we identify ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ with

$$ \mathbf{k}[x_{\mathbf{i}}(1),x_{\mathbf{i}}(2),\dots x_{\mathbf{i}}(m)]={}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} $$

via (4.5). The later is given the obvious $\mathbf{k}$-algebra structure.

4.15. Proposition. *For $\mathbf{i},\mathbf{i}',\mathbf{i}''$ in ${}^\theta I^\nu$ and f in ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ the following hold :*

(a) $1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}\star f=f$ *if* $\mathbf{i}=\mathbf{i}'$ *and* $1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}\star f=0$ *else.*

(b) $\varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'',\mathbf{i}'}(l)\star f=0$ *unless* $\mathbf{i}''=\mathbf{i}'=\mathbf{i}$ *and* $\varkappa_{\mathbf{\Lambda},\mathbf{V}}(l)\star f=x_{\mathbf{i}}(l)f$.

(c) $\pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'',\mathbf{i}'}(1)\star f=0$ *unless* $\mathbf{i}'=\mathbf{i}$, $\mathbf{i}''=\varepsilon_1\mathbf{i}$ *and*

$$ \pi_{\mathbf{\Lambda},\mathbf{V}}(1)\star f=x_{\varepsilon_1\mathbf{i}}(0)^{\lambda_{\varepsilon_1\mathbf{i}}(0)}\varepsilon_1(f). $$

(d) $\sigma_{\mathbf{V},\mathbf{i}'',\mathbf{i}'}(k)\star f=0$ *unless* $\mathbf{i}'=\mathbf{i}$ *and* $\mathbf{i}''=s_k\mathbf{i}$ *or* $\mathbf{i}$, *and we have*

- *if* $s_k\mathbf{i}=\mathbf{i}$ *and* $k\neq 0$ *then*

$$ \sigma_{\mathbf{V}}(k)\star f=(x_{\mathbf{i}}(k+1)-x_{\mathbf{i}}(k))^{h_{\mathbf{i}}(k)}(s_k(f)-f), $$

- *if* $s_0\mathbf{i}=\mathbf{i}$ *then*

$$ \sigma_{\mathbf{V}}(0)\star f=(x_{\mathbf{i}}(2)-x_{\mathbf{i}}(0))^{h_{\mathbf{i}}(0)}x_{\mathbf{i}}(1)^{\lambda_{\mathbf{i}}(1)}x_{\mathbf{i}}(2)^{\lambda_{\mathbf{i}}(2)}(s_0(f)-f), $$

- *if* $s_k\mathbf{i}\neq\mathbf{i}$ *and* $k\neq 0$ *then*

$$\sigma_{\mathbf{V},s_k\mathbf{i},\mathbf{i}}(k)\star f=(x_{s_k\mathbf{i}}(k+1)-x_{s_k\mathbf{i}}(k))^{h_{s_k\mathbf{i}}(k)}s_k(f),$$

$$\sigma_{\mathbf{V},\mathbf{i},\mathbf{i}}(k)\star f=(x_{\mathbf{i}}(k+1)-x_{\mathbf{i}}(k))^{h_{\mathbf{i}}(k)}f$$

- *if* $s_0\mathbf{i}\neq\mathbf{i}$ *then*

$$\sigma_{\mathbf{V},s_0\mathbf{i},\mathbf{i}}(0)\star f=(x_{s_0\mathbf{i}}(2)-x_{s_0\mathbf{i}}(0))^{h_{s_0\mathbf{i}}(0)}x_{s_0\mathbf{i}}(1)^{\lambda_{s_0\mathbf{i}}(1)}x_{s_0\mathbf{i}}(2)^{\lambda_{s_0\mathbf{i}}(2)}s_0(f),$$

$$\sigma_{\mathbf{V},\mathbf{i},\mathbf{i}}(0)\star f=(x_{\mathbf{i}}(2)-x_{\mathbf{i}}(0))^{h_{\mathbf{i}}(0)}x_{\mathbf{i}}(1)^{\lambda_{\mathbf{i}}(1)}x_{\mathbf{i}}(2)^{\lambda_{\mathbf{i}}(2)}f.$$

Proof : Parts (a), (b) are left to the reader. Let $w\in W_{\mathbf{i}}$. Recall that ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}\subset{}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ and that ψ_w lies in ${}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$. Under the map (4.6) the multiplication in ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ and the $\mathbf{S}$-action on ${}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ are related by the following formula

$$f(-x_{\mathbf{i}}(1),\dots,-x_{\mathbf{i}}(m))\,\psi_w=w(f)\,\psi_w.$$

Further we have $\varepsilon_1(\mathbf{i}_w)=\mathbf{i}_{w\varepsilon_1}$. Therefore, part (c) follows from the following computation, see Proposition 4.13$(c),(d),(f)$,

$$[{}^\theta Z^{\varepsilon_1}_{\mathbf{\Lambda},\mathbf{V}}]\star\psi_w=\Lambda^{\varepsilon_1}_{w\varepsilon_1,w}\Lambda_w\psi_{w\varepsilon_1}=(\chi_{w\varepsilon_1(1)})^{\lambda_{i_{w\varepsilon_1(0)}}}\psi_{w\varepsilon_1}=(-\chi_{w\varepsilon_1(0)})^{\lambda_{i_{w\varepsilon_1(0)}}}\psi_{w\varepsilon_1},$$

where $i_{w\varepsilon_1(l)}$ is the l-th component of the sequence $\mathbf{i}_{w\varepsilon_1}$. Let us concentrate on (d). The first claim is obvious because ${}^\theta Z^{s_k}_{w,w'}=\emptyset$ unless $w'=w,ws_k$ by Lemma 4.6, and $\mathbf{i}_{ws_k}=s_k\mathbf{i}_w$. Now, given $\mathbf{i}'=\mathbf{i}$ or $s_k\mathbf{i}$ we must compute the linear operator

$$(4.7)\qquad {}^\theta\mathcal{F}_{\mathbf{V},\mathbf{i}}\to{}^\theta\mathcal{F}_{\mathbf{V},\mathbf{i}'},\quad f\mapsto\sigma_{\mathbf{V},\mathbf{i}',\mathbf{i}}(k)\star f.$$

Proposition 4.13(b) yields an embedding

$${}^\theta\mathcal{F}_{\mathbf{V},\mathbf{i}}\to\bigoplus_{w\in W_{\mathbf{i}}}\mathbf{Q}\psi_w,\quad f(-x_{\mathbf{i}}(1),\dots,-x_{\mathbf{i}}(m))\mapsto\sum_{w\in W_{\mathbf{i}}}w(f)\Lambda_w^{-1}\psi_w.$$

Under this inclusion the map (4.7) is of the following form

$$\sum_{w\in W_{\mathbf{i}}}w(f)\Lambda_w^{-1}\psi_w\mapsto\sum_{w'\in W_{\mathbf{i}'}}g_{w'}\psi_{w'},\quad g_{w'}=\sum_{w\in W_{\mathbf{i}}}w(f)\Lambda^{s_k}_{w',w}$$

by Proposition 4.13$(c),(d)$. We claim that the right hand side is the image of a polynomial g in ${}^\theta\mathcal{F}_{\mathbf{V},\mathbf{i}'}$ that we'll compute explicitly. The polynomial g is completely determined by the following relations

$$(4.8)\qquad g_{w'}=w'(g)\Lambda_{w'}^{-1},\quad\forall w'\in W_{\mathbf{i}'}.$$

In the rest of the proof we'll fix w,w' in the following way

$$w\in W_{\mathbf{i}},\quad w'\in W_{\mathbf{i}'},\quad w'=w\text{ or }ws_k.$$

In particular we have $\mathbf{i} = \mathbf{i}_w$, $\mathbf{i}' = \mathbf{i}_{w'}$, and $\mathbf{i}' = \mathbf{i}$ or $s_k\mathbf{i}$.

(*i*) First, assume that $s_k\mathbf{i} = \mathbf{i}$. Then $\mathbf{i}' = \mathbf{i}$, $w's_k \in W_{w'}$, and we have

$$g_{w'} = w'(f)\Lambda^{s_k}_{w',w'} + w's_k(f)\Lambda^{s_k}_{w',w's_k}.$$

Section 4.9 and Proposition 4.13 yield

$$\Lambda_{w'} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w'} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w'}),$$

$$\Lambda^{s_k}_{w',w'} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w'} \oplus {}^\theta\mathfrak{m}_{\mathbf{V},w',w's_k})^{-1},$$

$$\Lambda^{s_k}_{w',w's_k} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w'} \oplus {}^\theta\mathfrak{m}_{\mathbf{V},w's_k,w'})^{-1},$$

$$\mathrm{eu}({}^\theta\mathfrak{m}_{\mathbf{V},w',w's_k}) = -\mathrm{eu}({}^\theta\mathfrak{m}_{\mathbf{V},w's_k,w'}) = w'(\alpha_k).$$

So we have

$$\Lambda^{s_k}_{w',w'} = \mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k})w'(\alpha_k)^{-1}\Lambda^{-1}_{w'} = -\Lambda^{s_k}_{w',w's_k}.$$

Therefore we obtain

$$g_{w'} = w'(f - s_k(f))\,\mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k})w'(\alpha_k)^{-1}\Lambda^{-1}_{w'}.$$

Now, assume that $k \neq 0$. There is no arrow joining $i_{w'(k)}$ and $i_{w'(k+1)}$, because $i_{w'(k)} = i_{w'(k+1)}$. Thus Section 4.9 yields

$$\mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k}) = 1.$$

Hence

$$\begin{aligned} g_{w'} &= w'(f - s_k(f))w'(\alpha_k)^{-1}\Lambda^{-1}_{w'} \\ &= w'(g)\Lambda^{-1}_{w'}, \\ g &= (f - s_k(f))\alpha_k^{-1}. \end{aligned}$$

Next, assume that $k = 0$. There is no arrow joining $i_{w'(0)}$ and $i_{w'(2)}$. Thus Section 4.9 yields

$$\mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_0}) = (-\chi_{w'(1)})^{\lambda_{i_{w'(1)}}}(-\chi_{w'(2)})^{\lambda_{i_{w'(2)}}}.$$

Therefore we have

$$\begin{aligned} g_{w'} &= w'(f - s_0(f))(-\chi_{w'(1)})^{\lambda_{i_{w'(1)}}}(-\chi_{w'(2)})^{\lambda_{i_{w'(2)}}}\,w'(\alpha_0)^{-1}\Lambda^{-1}_{w'} \\ &= w'(g)\Lambda^{-1}_{w'}, \\ g &= (f - s_0(f))(-\chi_1)^{\lambda_{i_{w'(1)}}}(-\chi_2)^{\lambda_{i_{w'(2)}}}\,\alpha_0^{-1}. \end{aligned}$$

(*ii*) Finally, assume that $s_k\mathbf{i} \neq \mathbf{i}$, i.e., that $ws_k \notin W_w$. Section 4.9 and Proposition 4.13 yield

$$\mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},ws_k}) = \mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},w}),$$

$$\Lambda_w = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w}),$$

$$\Lambda_{ws_k} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},ws_k} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w}),$$

$$\Lambda^{s_k}_{w,w} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w,ws_k} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w})^{-1} = \Lambda^{s_k}_{ws_k,w}.$$

So we have

$$\Lambda^{s_k}_{ws_k,w}\Lambda_{ws_k} = \mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},ws_k,w}),$$

$$\Lambda^{s_k}_{w,w}\Lambda_{w} = \mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w,ws_k}).$$

Next, one of the two following alternatives holds :

- either $\mathbf{i}' = s_k\mathbf{i}$, $w' = ws_k$ and

$$\begin{aligned} g_{w'} &= w's_k(f)\Lambda^{s_k}_{w',w's_k} \\ &= w's_k(f)(\Lambda^{s_k}_{ws_k,w}\Lambda_{ws_k})\Lambda^{-1}_{w'} \\ &= w's_k(f)\,\mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k})\,\Lambda^{-1}_{w'}. \end{aligned}$$

- or $\mathbf{i}' = \mathbf{i}$, $w' = w$ and

$$\begin{aligned} g_{w'} &= w'(f)\Lambda^{s_k}_{w',w'} \\ &= w'(f)(\Lambda^{s_k}_{w,w}\Lambda_{w})\Lambda^{-1}_{w'} \\ &= w'(f)\,\mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k})\,\Lambda^{-1}_{w'}. \end{aligned}$$

Now we consider the cases $k \neq 0$ and $k = 0$. First, assume that $k \neq 0$. By Section 4.9 we have

$$\mathrm{eu}({}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w',w's_k}) = w'(\alpha_k)^{h_{i_{w'(k)},i_{w'(k+1)}}}.$$

Thus (4.8) holds with

$$g = s_k(f)(-\alpha_k)^{h_{i_{w'(k)},i_{w'(k+1)}}}$$

in the first case and with

$$g = f(-\alpha_k)^{h_{i_{w'(k)},i_{w'(k+1)}}}$$

in the second one. Next, assume that $k = 0$. By Section 4.9 we have

$$\mathrm{eu}({}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w',w's_0}) = w'(\chi_1)^{\lambda_{i_{w'(1)}}}\,w'(\chi_2)^{\lambda_{i_{w'(2)}}}\,w'(\alpha_0)^{h_{i_{w'(0)},i_{w'(2)}}}.$$

Thus (4.8) holds with

$$g = s_0(f)(-\chi_1)^{\lambda_{i_{w'(1)}}}\,(-\chi_2)^{\lambda_{i_{w'(2)}}}\,(-\alpha_0)^{h_{i_{w'(0)},i_{w'(2)}}}$$

in the first case and with

$$g = f(-\chi_1)^{\lambda_{i_{w'(1)}}}\,(-\chi_2)^{\lambda_{i_{w'(2)}}}\,(-\alpha_0)^{h_{i_{w'(0)},i_{w'(2)}}}$$

in the second one.

$\square$

4.16. Description of the graded k-algebra ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$. We can use the previous computations concerning ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$ to get informations on ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$. The action of

$$1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, \quad \varkappa_{\mathbf{\Lambda},\mathbf{V}}(l), \quad \sigma_{\mathbf{\Lambda},\mathbf{V}}(k), \quad \pi_{\mathbf{\Lambda},\mathbf{V}}(1),$$

yields linear operators in $\mathrm{End}({}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}})$. Recall that ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$ is a faithful left ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$-module and that there are canonical isomorphisms

$${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}} = {}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}, \quad {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}} = {}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}.$$

Thus the graded left ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$-module ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ is also faithful. Recall also that

$${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}} = \bigoplus_{\mathbf{i}\in{}^\theta I^\nu} {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, \quad {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} = \mathbf{k}[x_{\mathbf{i}}(1), x_{\mathbf{i}}(2), \dots, x_{\mathbf{i}}(m)].$$

We obtain the following.

4.17. Theorem. *The graded* $\mathbf{k}$*-algebra* ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$ *is isomorphic to a graded* $\mathbf{k}$*-subalgebra of* $\mathrm{End}({}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}})$ *which contains the linear operators*

$$1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, \quad \varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l), \quad \sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(k), \quad \pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(1),$$

$$\mathbf{i}\in{}^\theta I^\nu, \quad k=0,1,\dots,m-1, \quad l=1,2,\dots,m,$$

defined as follows :

(*a*) $1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ *is the projection to* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ *relatively to* $\bigoplus_{\mathbf{i}'\neq\mathbf{i}}{}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}$*,*

(*b*) $\varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l)=0$ *on* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}$ *if* $\mathbf{i}'\neq\mathbf{i}$*, and it acts by multiplication by* $x_{\mathbf{i}}(l)$ *on* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$*,*

(*c*) $\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(k)=0$ *on* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}$ *if* $\mathbf{i}'\neq\mathbf{i}$*, and it takes a polynomial* f *in* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ *to*

$$\begin{aligned}
&(x_{\mathbf{i}}(k+1)-x_{\mathbf{i}}(k))^{h_{\mathbf{i}}(k)}(s_k(f)-f) && \text{if } s_k\mathbf{i}=\mathbf{i}\,, k\neq 0,\\
&(x_{\mathbf{i}}(2)-x_{\mathbf{i}}(0))^{h_{\mathbf{i}}(0)}x_{\mathbf{i}}(1)^{\lambda_{\mathbf{i}}(1)}x_{\mathbf{i}}(2)^{\lambda_{\mathbf{i}}(2)}(s_0(f)-f) && \text{if } s_k\mathbf{i}=\mathbf{i}\,, k=0,\\
&(x_{s_k\mathbf{i}}(k+1)-x_{s_k\mathbf{i}}(k))^{h_{s_k\mathbf{i}}(k)}s_k(f) && \text{if } s_k\mathbf{i}\neq\mathbf{i}\,, k\neq 0,\\
&(x_{s_0\mathbf{i}}(2)-x_{s_0\mathbf{i}}(0))^{h_{s_0\mathbf{i}}(0)}x_{s_0\mathbf{i}}(1)^{\lambda_{s_0\mathbf{i}}(1)}x_{s_0\mathbf{i}}(2)^{\lambda_{s_0\mathbf{i}}(2)}s_0(f) && \text{if } s_k\mathbf{i}\neq\mathbf{i},\ k=0,
\end{aligned}$$

(*d*) $\pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(1)=0$ *on* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}$ *if* $\mathbf{i}'\neq\mathbf{i}$*, and it takes a polynomial* f *in* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ *to*

$$x_{\varepsilon_1\mathbf{i}}(0)^{\lambda_{\varepsilon_1\mathbf{i}}(0)}\varepsilon_1(f).$$

The degrees of these operators are given by the following formulas

$$\begin{aligned}
&\deg(1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}})=0,\\
&\deg(\varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l))=2,\\
&\deg(\pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(1))=2\lambda_{\varepsilon_1\mathbf{i}}(0),\\
&\deg(\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(0))=2h_{s_0\mathbf{i}}(0)+2\lambda_{s_0\mathbf{i}}(1)+2\lambda_{s_0\mathbf{i}}(2),\\
&\deg(\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(k))=2h_{s_k\mathbf{i}}(k) \quad \text{if } k\neq 0.
\end{aligned}$$

4.18. Shift of the grading. We are mostly interested by the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$, whose grading differs from the grading of ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$. Let us compute the degree of the generators of ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$. We have

$$ {}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V},s_k\mathbf{i},\mathbf{i}} = {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},s_k\mathbf{i},\mathbf{i}}[d_{\lambda,\mathbf{i}} - d_{\lambda,s_k\mathbf{i}}]. $$

Recall that $h_{\theta(i),j} = h_{\theta(j),i}$ for each i,j. Hence, an easy computation using Proposition 2.5 yields

$$ d_{\lambda,\mathbf{i}} - d_{\lambda,s_k\mathbf{i}} = \begin{cases} h_\mathbf{i}(k) - h_{s_k\mathbf{i}}(k) & \text{if } k \neq 0, \\ h_\mathbf{i}(0) - h_{s_0\mathbf{i}}(0) + \lambda_\mathbf{i}(2) + \lambda_\mathbf{i}(1) - \lambda_{s_0\mathbf{i}}(2) - \lambda_{s_0\mathbf{i}}(1) & \text{if } k = 0, \end{cases} $$

$$ d_{\lambda,\mathbf{i}} - d_{\lambda,\varepsilon_1\mathbf{i}} = \lambda_{\varepsilon_1\mathbf{i}}(0) - \lambda_\mathbf{i}(0). $$

Therefore the grading of ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$ is given by the following rules :

$$ \begin{aligned} &\deg(1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}) = 0, \\ &\deg(\varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l)) = 2, \\ &\deg(\pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(1)) = \lambda_\mathbf{i}(0) + \lambda_\mathbf{i}(1), \\ &\deg(\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(0)) = -i_0 \cdot i_2 + \lambda_\mathbf{i}(-1) + \lambda_\mathbf{i}(0) + \lambda_\mathbf{i}(1) + \lambda_\mathbf{i}(2), \\ &\deg(\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(k)) = -i_k \cdot i_{k+1} \quad \text{if } k \neq 0. \end{aligned} $$

5. The graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector $\nu \neq 0$ in ${}^\theta\mathbb{N}I$ and a dimension vector λ in $\mathbb{N}I$. Set $|\nu| = 2m$.

5.1. Definition of the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. Assume that $m > 0$. We define a graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ with 1 generated by $1_\mathbf{i}$, $\varkappa_l$, σ_k, π_1 with $\mathbf{i} = (i_{1-m},\dots,i_m)$ in ${}^\theta I^\nu$, $k = 1,\dots,m-1$, $l = 1,2,\dots,m$, modulo the following defining relations†

(a) $1_\mathbf{i}1_{\mathbf{i}'} = \delta_{\mathbf{i},\mathbf{i}'}1_\mathbf{i}$, $\quad \sigma_k 1_\mathbf{i} = 1_{s_k\mathbf{i}}\sigma_k$, $\quad \varkappa_l 1_\mathbf{i} = 1_\mathbf{i}\varkappa_l$, $\quad \pi_1 1_\mathbf{i} = 1_{\varepsilon_1\mathbf{i}}\pi_1$,

(b) $\varkappa_l\varkappa_{l'} = \varkappa_{l'}\varkappa_l$, $\quad \pi_1\varkappa_l = \varkappa_{\varepsilon_1(l)}\pi_1$,

(c) $\sigma_k^2 1_\mathbf{i} = Q_{i_k,i_{k+1}}(\varkappa_{k+1},\varkappa_k)1_\mathbf{i}$, $\quad \pi_1^2 1_\mathbf{i} = \varkappa_0^{\lambda_{i_0}}\varkappa_1^{\lambda_{i_1}}1_\mathbf{i}$,

(d) $\sigma_k\sigma_{k'} = \sigma_{k'}\sigma_k$ if $k \neq k' \pm 1$, $\quad \pi_1\sigma_k = \sigma_k\pi_1$ if $k \neq 1$,

(e) $(\sigma_1\pi_1)^2 1_\mathbf{i} = (\pi_1\sigma_1)^2 1_\mathbf{i} + \delta_{i_0,i_2}(-1)^{\lambda_{i_2}} \dfrac{\varkappa_0^{\lambda_{i_1}+\lambda_{i_2}} - \varkappa_2^{\lambda_{i_1}+\lambda_{i_2}}}{\varkappa_0 - \varkappa_2}\sigma_1 1_\mathbf{i}$,

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†We thank M. Kashiwara who indicate us an error in a previous version of the relations

(f) $(\sigma_{k+1}\sigma_k\sigma_{k+1} - \sigma_k\sigma_{k+1}\sigma_k)1_{\mathbf{i}} =$

$$= \delta_{i_k,i_{k+2}} \frac{Q_{i_k,i_{k+1}}(\varkappa_{k+1},\varkappa_k) - Q_{i_k,i_{k+1}}(\varkappa_{k+1},\varkappa_{k+2})}{\varkappa_k - \varkappa_{k+2}} 1_{\mathbf{i}},$$

$$(g)\ \ (\sigma_k\varkappa_l - \varkappa_{s_k(l)}\sigma_k)1_{\mathbf{i}} = \begin{cases} -1_{\mathbf{i}} & \text{if } l = k,\ i_k = i_{k+1}, \\ 1_{\mathbf{i}} & \text{if } l = k+1,\ i_k = i_{k+1}, \\ 0 & \text{else.} \end{cases}$$

Here $\delta_{i,j}$ is the Kronecker symbol, $\varkappa_{1-l} = -\varkappa_l$, and

$$Q_{i,j}(u,v) = \begin{cases} (-1)^{h_{i,j}}(u-v)^{-i\cdot j} & \text{if } i \neq j, \\ 0 & \text{else.} \end{cases} \tag{5.1}$$

We'll abbreviate $\sigma_{\mathbf{i},k} = \sigma_k 1_{\mathbf{i}}$, $\varkappa_{\mathbf{i},l} = \varkappa_l 1_{\mathbf{i}}$, and $\pi_{\mathbf{i},1} = \pi_1 1_{\mathbf{i}}$. The grading on ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ is given by the following rules :

$$\deg(1_{\mathbf{i}}) = 0,$$

$$\deg(\varkappa_{\mathbf{i},l}) = 2,$$

$$\deg(\pi_{\mathbf{i},1}) = \lambda_{i_0} + \lambda_{i_1},$$

$$\deg(\sigma_{\mathbf{i},k}) = -i_k \cdot i_{k+1}.$$

If $\nu = 0$ we set ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} = \mathbf{k}$ as a graded $\mathbf{k}$-algebra. Let ω be the unique anti-involution of the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ which fixes $1_{\mathbf{i}}$, $\varkappa_l$, σ_k, π_1.

5.2. Remarks. (a) We may set $\sigma_0 = \pi_1\sigma_1\pi_1$. We have

$$\deg(\sigma_0 1_{\mathbf{i}}) = -i_0 \cdot i_2 + \lambda_{i_{-1}} + \lambda_{i_0} + \lambda_{i_1} + \lambda_{i_2}.$$

(b) We may also set $\pi_l = \sigma_{l-1}\dots\sigma_2\sigma_1\pi_1\sigma_1\sigma_2\dots\sigma_{l-1}$. We have

$$\deg(\pi_l 1_{\mathbf{i}}) = -(i_1 + i_2 + \cdots + i_{l-1})\cdot(i_l + i_{1-l}) + \lambda_{i_l} + \lambda_{i_{1-l}}.$$

5.3. The polynomial representation and the PBW theorem. Given any objects $\mathbf{V}$ in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ and $\boldsymbol{\Lambda}$ in $\boldsymbol{\mathcal{V}}_\lambda$ we abbreviate

$${}^\theta\mathbf{F}_\nu = {}^\theta\mathbf{F}_{\boldsymbol{\Lambda},\mathbf{V}}, \quad {}^\theta\mathbf{F}_{\mathbf{i}} = {}^\theta\mathbf{F}_{\boldsymbol{\Lambda},\mathbf{V},\mathbf{i}}, \quad {}^\theta\mathbf{S}_\nu = {}^\theta\mathbf{S}_{\mathbf{V}}.$$

5.4. Proposition. *There is an unique graded* $\mathbf{k}$*-algebra morphism* ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} \to \mathrm{End}({}^\theta\mathbf{F}_\nu)$ *such that, for each* $\mathbf{i} \in {}^\theta I^\nu$, $k = 0,1,\dots,m-1$, $l = 1,2,\dots,m$, *we have*

$$1_{\mathbf{i}} \mapsto 1_{\boldsymbol{\Lambda},\mathbf{V},\mathbf{i}}, \quad \varkappa_{\mathbf{i},l} \mapsto \varkappa_{\boldsymbol{\Lambda},\mathbf{V},\mathbf{i}}(l), \quad \sigma_{\mathbf{i},k} \mapsto \sigma_{\boldsymbol{\Lambda},\mathbf{V},\mathbf{i}}(k), \quad \pi_{\mathbf{i},1} \mapsto \pi_{\boldsymbol{\Lambda},\mathbf{V},\mathbf{i}}(1).$$

Proof : The defining relations of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ are checked by a direct computation. Let us (only) give a few indications concerning the relation 5.1(e). We have

$$\sigma_1 1_{\mathbf{i}} = (\varkappa_1 - \varkappa_2)^{h_{s_1\mathbf{i}}(1)}(s_1 - \delta_{i_1,i_2})1_{\mathbf{i}}, \quad \pi_1 1_{\mathbf{i}} = \varkappa_0^{\lambda_{i_1}}\varepsilon_1 1_{\mathbf{i}}.$$

This yields

$$\sigma_1\pi_1 1_{\mathbf{i}} = (\varkappa_1 - \varkappa_2)^{h_{s_0}\mathbf{i}(0)}(s_1 - \delta_{i_0,i_2})\varkappa_0^{\lambda_{i_1}}\varepsilon_1 1_{\mathbf{i}},$$

$$\pi_1\sigma_1 1_{\mathbf{i}} = \varkappa_0^{\lambda_{i_2}}\varepsilon_1(\varkappa_1 - \varkappa_2)^{h_{s_1}\mathbf{i}(1)}(s_1 - \delta_{i_1,i_2})1_{\mathbf{i}}.$$

Therefore we have

$$(\sigma_1\pi_1)^2 1_{\mathbf{i}} = (\varkappa_1 - \varkappa_2)^{h_{s_1}\mathbf{i}(1)}(s_1 - \delta_{i_1,i_2})\varkappa_0^{\lambda_{i_2}}\varepsilon_1(\varkappa_1 - \varkappa_2)^{h_{s_0}\mathbf{i}(0)}(s_1 - \delta_{i_0,i_2})\varkappa_0^{\lambda_{i_1}}\varepsilon_1 1_{\mathbf{i}},$$

$$(\pi_1\sigma_1)^2 1_{\mathbf{i}} = \varkappa_0^{\lambda_{i_1}}\varepsilon_1(\varkappa_1 - \varkappa_2)^{h_{s_0}\mathbf{i}(0)}(s_1 - \delta_{i_0,i_2})\varkappa_0^{\lambda_{i_2}}\varepsilon_1(\varkappa_1 - \varkappa_2)^{h_{s_1}\mathbf{i}(1)}(s_1 - \delta_{i_1,i_2})1_{\mathbf{i}}.$$

Hence we have

$$(\sigma_1\pi_1)^2 1_{\mathbf{i}} = (\varkappa_1 - \varkappa_2)^{h_{s_1}\mathbf{i}(1)}(\varkappa_0 - \varkappa_2)^{h_{s_0}\mathbf{i}(0)}A,$$

$$(\pi_1\sigma_1)^2 1_{\mathbf{i}} = (\varkappa_1 - \varkappa_2)^{h_{s_1}\mathbf{i}(1)}(\varkappa_0 - \varkappa_2)^{h_{s_0}\mathbf{i}(0)}B,$$

where

$$A = (s_1 - \delta_{i_1,i_2})\varkappa_0^{\lambda_{i_2}}\varepsilon_1(s_1 - \delta_{i_0,i_2})\varkappa_0^{\lambda_{i_1}}\varepsilon_1 1_{\mathbf{i}},$$

$$B = \varkappa_0^{\lambda_{i_1}}\varepsilon_1(s_1 - \delta_{i_0,i_2})\varkappa_0^{\lambda_{i_2}}\varepsilon_1(s_1 - \delta_{i_1,i_2})1_{\mathbf{i}}.$$

If $i_0 \neq i_2$ it is easy to see that $A = B$. If $i_0 = i_2$ a direct computation yields

$$B - A = \big(\varkappa_2^{\lambda_{i_1}}(-\varkappa_2)^{\lambda_{i_2}} - \varkappa_1^{\lambda_{i_2}}(-\varkappa_1)^{\lambda_{i_1}}\big)s_1.$$

The rest of the computation is left to the reader.

□

The $\mathbf{k}$-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ is a left graded ${}^\theta\mathbf{F}_\nu$-module such that $x_{\mathbf{i}}(l)$ acts by the left multiplication with the element $\varkappa_{\mathbf{i},l}$ for each $l = 1, 2, \ldots, m$. To unburden the notation we may write $\varkappa_{\mathbf{i},l} = x_{\mathbf{i}}(l)$. The following convention is important.

From now on we'll regard W_m as a Weyl group of type B_m, with the set of simple reflections $\{s_1, s_2, \ldots, s_m\}$ where $s_m = \varepsilon_1$, rather than an extended Weyl group of type D_m as in Section 4.2.

For $w \in W_m$ we choose a reduced decomposition $\dot{w}$ of w. By the observation above $\dot{w}$ is a minimal decomposition of the following form

$$w = s_{k_1}s_{k_2}\cdots s_{k_r}, \quad 0 < k_1, k_2, \ldots, k_r \leqslant m, \quad s_m = \varepsilon_1.$$

We define an element $\sigma_{\dot{w}}$ in ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ by the following formula

$$\text{(5.2)} \qquad \sigma_{\dot{w}} = \sum_{\mathbf{i}} 1_{\mathbf{i}}\sigma_{\dot{w}}, \quad 1_{\mathbf{i}}\sigma_{\dot{w}} = \begin{cases} 1_{\mathbf{i}} & \text{if } r = 0 \\ 1_{\mathbf{i}}\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_r} & \text{else,} \end{cases}$$

where we have set $\sigma_m = \pi_1$. Observe that $\sigma_{\dot{w}}$ may depend on the choice of the reduced decomposition $\dot{w}$.

5.5. Proposition. *The* $\mathbf{k}$*-algebra* ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ *is a free (left or right)* ${}^\theta\mathbf{F}_\nu$*-module with basis* $\{\sigma_{\dot w};\, w\in W_m\}$*. Its rank is* $2^m m!$*. The operator* $1_\mathbf{i}\sigma_{\dot w}$ *is homogeneous and its degree is independent of the choice of the reduced decomposition* $\dot w$*.*

Proof : The $\mathbf{k}$-space ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ is filtered with $1_\mathbf{i}$, $\varkappa_{\mathbf{i},l}$ in degree 0 and $\sigma_{\mathbf{i},k}$, $\pi_{\mathbf{i},1}$ in degree 1. This filtration is a nonnegative increasing $\mathbf{k}$-algebra filtration. Each term of the filtration is a graded subspace of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. Therefore the associated graded $\mathbf{k}$-algebra $\mathrm{gr}\,{}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ is bigraded and the symbol map preserves the grading.

Now, the *Nil Hecke algebra* of type B_m is the $\mathbf{k}$-algebra ${}^\theta\mathbf{NH}_m$ generated by the elements $\bar\pi_1,\bar\sigma_1,\bar\sigma_2,\dots,\bar\sigma_{m-1}$ with the relations

$$\bar\sigma_k\bar\sigma_{k'}=\bar\sigma_{k'}\bar\sigma_k \text{ if } |k-k'|>1,\quad \bar\pi_1\bar\sigma_k=\bar\sigma_k\bar\pi_1 \text{ if } k\neq 1,\quad (\bar\pi_1\bar\sigma_1)^2=(\bar\sigma_1\bar\pi_1)^2,$$

$$\bar\sigma_{k+1}\bar\sigma_k\bar\sigma_{k+1}=\bar\sigma_k\bar\sigma_{k+1}\bar\sigma_k,\quad \bar\pi_1^2=\bar\sigma_k^2=0.$$

We can form the semidirect product ${}^\theta\mathbf{F}_\nu\rtimes{}^\theta\mathbf{NH}_m$, which is generated by $1_\mathbf{i}$, $\bar\varkappa_l$, $\bar\pi_1,\bar\sigma_k$ with the relations above and

$$\bar\sigma_k\bar\varkappa_l=\bar\varkappa_{s_k(l)}\bar\sigma_k,\quad \bar\pi_1\bar\varkappa_l=\bar\varkappa_{\varepsilon_1(l)}\bar\pi_1\ ,\quad \bar\varkappa_l\bar\varkappa_{l'}=\bar\varkappa_{l'}\bar\varkappa_{l'}.$$

We have a surjective $\mathbf{k}$-algebra morphism

$$ {}^\theta\mathbf{F}_\nu\rtimes{}^\theta\mathbf{NH}_m\to\mathrm{gr}\,{}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu},\quad 1_\mathbf{i}\mapsto 1_\mathbf{i},\quad \bar\varkappa_l\mapsto\varkappa_l,\quad \bar\pi_1\mapsto\pi_1,\quad \bar\sigma_k\mapsto\sigma_k. \tag{5.3}$$

Thus the elements $\sigma_{\dot w}$ with $w\in W_m$ generate ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ as a ${}^\theta\mathbf{F}_\nu$-module. We must prove that they yield indeed a basis of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. This is rather clear, since the images of these elements in $\mathrm{End}({}^\theta\mathbf{F}_\nu)$ under the polynomial representation are independent over ${}^\theta\mathbf{F}_\nu$ (by Galois theory). Therefore the map (5.3) is invertible. The last claim is now clear, because the element $\sigma_{\dot w}$ has the same degree as its symbol and if $\dot w$, $\ddot w$ are two reduced decomposition of w then $\sigma_{\dot w}$ and $\sigma_{\ddot w}$ have the same symbol.

$\square$

Let ${}^\theta\mathbf{F}'_\nu=\bigoplus_\mathbf{i}{}^\theta\mathbf{F}'_\mathbf{i}$, where ${}^\theta\mathbf{F}'_\mathbf{i}$ is the localization of the ring ${}^\theta\mathbf{F}_\mathbf{i}$ with respect to the multiplicative system generated by

$$\{\varkappa_{\mathbf{i},l}\pm\varkappa_{\mathbf{i},l'};\ 1\leqslant l\neq l'\leqslant m\}\cup\{\varkappa_{\mathbf{i},l};\ l=1,2,\dots,m\}.$$

5.6. Corollary. *The polynomial representation of* ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ *on* ${}^\theta\mathbf{F}_\nu$ *is faithful. The inclusion of* ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ *into* $\mathrm{End}({}^\theta\mathbf{F}_\nu)$ *yields an isomorphism of* ${}^\theta\mathbf{F}'_\nu$*-algebras from* ${}^\theta\mathbf{F}'_\nu\otimes_{{}^\theta\mathbf{F}_\nu}{}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ *to* ${}^\theta\mathbf{F}'_\nu\rtimes W_m$*, such that for each* $\mathbf{i}$ *and each* $l=1,2,\dots,m$*,* $k=1,2,\dots,m-1$ *we have*

$$\begin{aligned}
&1_\mathbf{i}\mapsto 1_\mathbf{i},\\
&\varkappa_{\mathbf{i},l}\mapsto\varkappa_l 1_\mathbf{i},\\
&\pi_{\mathbf{i},1}\mapsto\varkappa_0^{\lambda_{i_1}}\varepsilon_1 1_\mathbf{i},\\
&\sigma_{\mathbf{i},k}\mapsto\begin{cases}(\varkappa_k-\varkappa_{k+1})^{-1}(s_k-1)1_\mathbf{i} & \text{if } i_k=i_{k+1},\\ (\varkappa_k-\varkappa_{k+1})^{h_{i_{k+1},i_k}}s_k 1_\mathbf{i} & \text{if } i_k\neq i_{k+1}.\end{cases}
\end{aligned}\tag{5.4}$$

Restricting the ${}^\theta\mathbf{F}_\nu$-action on ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ to the subalgebra ${}^\theta\mathbf{S}_\nu$ of ${}^\theta\mathbf{F}_\nu$ we get a structure of graded ${}^\theta\mathbf{S}_\nu$-algebra on ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$.

5.7. Proposition. *(a)* ${}^\theta\mathbf{S}_\nu$ *is isomorphic to the center of* ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$.

(b) ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ *is a free graded module over* ${}^\theta\mathbf{S}_\nu$ *of rank* $(2^m m!)^2$.

Proof : First we prove (a). Recall that

$${}^\theta\mathbf{S}_\nu = \mathbf{k}[\chi_1, \chi_2, \dots, \chi_m]^{W_\nu} = \Big(\bigoplus_{\mathbf{i}} \mathbf{k}[\varkappa_1, \varkappa_2, \dots, \varkappa_m]1_{\mathbf{i}}\Big)^{W_m}.$$

Given a sequence $\mathbf{i}$ in ${}^\theta I^\nu$ the assignment $x \mapsto x1_{\mathbf{i}}$ embeds ${}^\theta\mathbf{S}_\nu$ as a central subalgebra of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. We must check that this map surjects onto the center of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. This follows from Corollary 5.6. Part (b) follows from (a) and Proposition 5.5.

□

In Section 9 we'll prove the following theorem.

5.8. Theorem. *For any* $\nu \in {}^\theta\mathbb{N}I$, $\lambda \in \mathbb{N}I$ *there is an unique graded* ${}^\theta\mathbf{S}_{\mathbf{V}}$*-algebra isomorphism*

$$\Psi : {}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} \to {}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$$

which intertwines the representations of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ *and* ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$ *on* ${}^\theta\mathbf{F}_\nu$.

5.9. Examples. (a) If $m = 0$ then ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} = \mathbf{k}$ by definition.

(b) Assume that $m = 1$. Fix a vertex i in I and set $\nu = i + \theta(i)$. We have ${}^\theta I^\nu = \{\mathbf{i}, \theta(\mathbf{i})\}$ with $\mathbf{i} = i\theta(i)$ and $\theta(\mathbf{i}) = \theta(i)i$. We have

$${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} = (\mathbf{k}[\varkappa_1] \oplus \pi_1\mathbf{k}[\varkappa_1])1_{\mathbf{i}} \oplus (\mathbf{k}[\varkappa_1] \oplus \pi_1\mathbf{k}[\varkappa_1])1_{\theta(\mathbf{i})},$$

$$\pi_1\varkappa_1 1_{\mathbf{i}} = -\varkappa_1\pi_1 1_{\mathbf{i}}, \quad \pi_1\varkappa_1 1_{\theta(\mathbf{i})} = -\varkappa_1\pi_1 1_{\theta(\mathbf{i})},$$

$$\pi_1^2 1_{\theta(\mathbf{i})} = (-1)^{\lambda_{\theta(i)}}\varkappa_1^{\lambda_i+\lambda_{\theta(i)}}1_{\theta(\mathbf{i})}, \quad \pi_1^2 1_{\mathbf{i}} = (-1)^{\lambda_i}\varkappa_1^{\lambda_i+\lambda_{\theta(i)}}1_{\mathbf{i}}.$$

The inclusion ${}^\theta\mathbf{S}_\nu \subset {}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ is given by

$$\mathbf{k}[\chi] \to {}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}, \quad \chi \mapsto (\varkappa_1 1_{\mathbf{i}}, 0, -\varkappa_1 1_{\theta(\mathbf{i})}, 0).$$

6. Affine Hecke algebras of type B

6.1. Affine Hecke algebras of type B. Given a connected reductive group G we call *affine Hecke algebra of* G the Hecke algebra of the extended affine Weyl group $W \ltimes P$ where W is the Weyl group of (G, T), P is the group of characters of T, and T is a maximal torus of G. Fix p, q in $\mathbf{k}^\times$. For any integer $m \geqslant 0$ we define the affine Hecke algebra $\mathbf{H}_m$ of type B_m to be the affine Hecke algebra of $SO(2m+1)$. It admits the following presentation, see e.g., [Mc]. If $m > 0$ then $\mathbf{H}_m$ is the $\mathbf{k}$-algebra generated by

$$T_k, \quad X_l^{\pm 1}, \quad k = 0, 1, \dots, m-1, \quad l = 1, 2, \dots, m$$

satisfying the following defining relations :

(*a*) $X_l X_{l'} = X_{l'} X_l$,

(*b*) $(T_0T_1)^2 = (T_1T_0)^2$, $T_kT_{k-1}T_k = T_{k-1}T_kT_{k-1}$ if $k \neq 0, 1$, and $T_kT_{k'} = T_{k'}T_k$ if $|k - k'| \neq 1$,

(*c*) $T_0X_1^{-1}T_0 = X_1$, $T_kX_kT_k = X_{k+1}$ if $k \neq 0$, and $T_kX_l = X_lT_k$ if $l \neq k, k + 1$,

(*d*) $(T_k - p)(T_k + p^{-1}) = 0$ if $k \neq 0$, and $(T_0 - q)(T_0 + q^{-1}) = 0$.

If $m = 0$ then $\mathbf{H}_0 = \mathbf{k}$, the trivial $\mathbf{k}$-algebra. Note that $\mathbf{H}_1$ is the $\mathbf{k}$-algebra generated by T_0, $X_1^{\pm 1}$ with the defining relations

$$T_0X_1^{-1}T_0 = X_1, \quad (T_0 - q)(T_0 + q^{-1}) = 0.$$

6.2. Intertwiners and blocks of $\mathbf{H}_m$. We define

$$\mathbf{A} = \mathbf{k}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_m^{\pm 1}], \quad \mathbf{A}' = \mathbf{A}[\Sigma^{-1}], \quad \mathbf{H}'_m = \mathbf{A}' \otimes_{\mathbf{A}} \mathbf{H}_m,$$

where Σ is the multiplicative set generated by

$$1 - X_lX_{l'}^{\pm 1}, \quad 1 - p^2X_lX_{l'}^{\pm 1}, \quad 1 - X_l^2, \quad 1 - q^2X_l^{\pm 2}, \quad l \neq l'.$$

For $k = 0, \ldots, m - 1$ the intertwiner φ_k in $\mathbf{H}'_m$ is given by the following formulas

$$\begin{aligned} \varphi_k - 1 &= \frac{X_k - X_{k+1}}{pX_k - p^{-1}X_{k+1}} (T_k - p) \quad \text{if } k \neq 0, \\ \varphi_0 - 1 &= \frac{X_1^{-2} - 1}{qX_1^{-2} - q^{-1}} (T_0 - q). \end{aligned} \tag{6.1}$$

The group W_m acts on $\mathbf{A}'$ as follows

$$(s_ka)(X_1, \ldots, X_m) = a(X_1, \ldots, X_{k+1}, X_k, \ldots, X_m),$$

$$(\varepsilon_1a)(X_1, \ldots, X_m) = a(X_1^{-1}, X_2, \ldots, X_m).$$

There is an isomorphism of $\mathbf{A}'$-algebras

$$\mathbf{A}' \rtimes W_m \to \mathbf{H}'_m, \quad s_k \mapsto \varphi_k, \quad \varepsilon_1 \mapsto \varphi_0, \quad k \neq 0.$$

The semi-direct product group $\mathbb{Z}\rtimes\mathbb{Z}_2 = \mathbb{Z}\rtimes\{-1, 1\}$ acts on $\mathbf{k}^\times$ by $(n, \varepsilon) : z \mapsto z^\varepsilon p^{2n}$. Given a $\mathbb{Z} \rtimes \mathbb{Z}_2$-invariant subset I of $\mathbf{k}^\times$ we denote by $\mathbf{H}_m$-$\mathbf{Mod}_I$ the category of all finitely generated $\mathbf{H}_m$-modules such that the action of $X_1, X_2, \ldots, X_m$ is locally finite and all the eigenvalues belong to I. We associate to the set I the quiver Γ with set of vertices I and with one arrow $p^2i \to i$ whenever i lies in I. We equip Γ with an involution θ such that $\theta(i) = i^{-1}$ for each vertex i and such that θ takes the arrow $p^2i \to i$ to the arrow $i^{-1} \to p^{-2}i^{-1}$. *We'll assume that the set I does not contain* 1 *nor* -1 *and that* $p \neq 1, -1$. Thus the involution θ has no fixed points and no arrow may join a vertex of Γ to itself.

6.3. Remark. We may assume that either I is a $\mathbb{Z}$-orbit or I contains at least one of $\pm q$, see the discussion in [EK1]. Thus, we can assume that one of the following two cases holds :

(a) I is a $\mathbb{Z}$-orbit which does not contain 1, -1, q, $-q$. So either $I = \{p^n;\, n \in \mathbb{Z}_{\mathrm{odd}}\}$ or $I = \{-p^n;\, n \in \mathbb{Z}_{\mathrm{odd}}\}$. Then Γ is of type A_∞ if p has infinite order and Γ is of type $\mathrm{A}_r^{(1)}$ if p^2 is a primitive r-th root of unity.

(b) $q \in I$ (the case $-q \in I$ is similar) and $-1, 1 \notin I$. Then we have $I = \{qp^{2n};\, n \in \mathbb{Z}\} \cup \{q^{-1}p^{2n};\, n \in \mathbb{Z}\}$ with $q^2 \neq p^{4n}$ for all $n \in \mathbb{Z}$. Thus Γ is of type A_∞, $\mathrm{A}_\infty \times \mathrm{A}_\infty$, $\mathrm{A}_r^{(1)}$, or $\mathrm{A}_r^{(1)} \times \mathrm{A}_r^{(1)}$.

6.4. $\mathbf{H}_m$-modules versus ${}^\theta\mathbf{R}_m$-modules. Given an element λ of $\mathbb{N}I$ we define the graded $\mathbf{k}$-algebra

$$ {}^\theta\mathbf{R}_{I,\lambda,m} = \bigoplus_\nu {}^\theta\mathbf{R}_{I,\lambda,\nu}, \quad {}^\theta\mathbf{R}_{I,\lambda,\nu} = {}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}, \quad {}^\theta I^m = \coprod_\nu {}^\theta I^\nu, $$

where ν runs over the set of all dimension vectors in ${}^\theta\mathbb{N}I$ such that $|\nu| = 2m$. When there is no risk of confusion we abbreviate ${}^\theta\mathbf{R}_m = {}^\theta\mathbf{R}_{I,\lambda,m}$ and ${}^\theta\mathbf{R}_\nu = {}^\theta\mathbf{R}_{I,\lambda,\nu}$. Note that the $\mathbf{k}$-algebra ${}^\theta\mathbf{R}_m$ may not have 1, because the set I may be infinite, and that ${}^\theta\mathbf{R}_0 = \mathbf{k}$ as a graded $\mathbf{k}$-algebra. From now on, unless specified otherwise we'll set

$$ \lambda = \sum_i i, \quad i \in I \cap \{q, -q\}. \tag{6.2} $$

Given sequences

$$ \mathbf{i} = (i_{1-m}, \dots, i_{m-1}, i_m), \quad \mathbf{i}' = (i'_1, \dots, i'_{m'-1}, i'_{m'}), $$

we define a sequence $\theta(\mathbf{i}')\mathbf{i}\mathbf{i}'$ as follows

$$ \theta(\mathbf{i}')\mathbf{i}\mathbf{i}' = (\theta(i'_{m'}), \dots, \theta(i'_1), i_{1-m}, \dots, i_m, i'_1, \dots, i'_{m'}). $$

Let ν, ν' be dimension vectors in ${}^\theta\mathbb{N}I$ and $\mathbb{N}I$ respectively such that $|\nu| = 2m$, $|\nu'| = m'$, and $m + m' = m''$. We define an idempotent in ${}^\theta\mathbf{R}_{m''}$ by

$$ 1_{\nu,\nu'} = \sum_{\mathbf{i},\mathbf{i}'} 1_{\theta(\mathbf{i}')\mathbf{i}\mathbf{i}'}, \quad \mathbf{i} \in {}^\theta I^\nu,\ \mathbf{i}' \in I^{\nu'}. $$

For $\nu'_1, \nu'_2, \dots, \nu'_r$ in $\mathbb{N}I$ we define $1_{\nu,\nu'_1,\dots,\nu'_r}$ in the same way. Finally, for any graded ${}^\theta\mathbf{R}_{m''}$-module M we set

$$ 1_{m,\nu'}M = \bigoplus_{\mathbf{i},\mathbf{i}'} 1_{\theta(\mathbf{i}')\mathbf{i}\mathbf{i}'}M, \quad \mathbf{i} \in {}^\theta I^m,\ \mathbf{i}' \in I^{\nu'}. \tag{6.3} $$

If M is a right graded ${}^\theta\mathbf{R}_{m''}$-module we define $M1_{m,\nu'}$ in the same way.

Next, let ${}^\theta\mathbf{R}_m$-$\mathbf{Mod}_0$ be the category of all finitely generated (non-graded) ${}^\theta\mathbf{R}_m$-modules such that the elements $\varkappa_1, \varkappa_2, \dots, \varkappa_m$ act locally nilpotently. Let

$$ {}^\theta\mathbf{R}_m\text{-}\mathbf{fMod}_0 \subset {}^\theta\mathbf{R}_m\text{-}\mathbf{Mod}_0, \quad \mathbf{H}_m\text{-}\mathbf{fMod}_I \subset \mathbf{H}_m\text{-}\mathbf{Mod}_I $$

be the full subcategories of finite dimensional modules.

Fix a formal series $f(\varkappa)$ in $\mathbf{k}[[\varkappa]]$ such that $f(\varkappa) = 1 + \varkappa$ modulo $(\varkappa^2)$.

6.5. Theorem. *We have an equivalence of categories*

$$^{\theta}\mathbf{R}_m\text{-}\mathbf{Mod}_0 \to \mathbf{H}_m\text{-}\mathbf{Mod}_I, \quad M \mapsto M$$

which is given by

(a) X_l *acts on* $1_{\mathbf{i}}M$ *by* $i_l^{-1}f(\varkappa_l)$ *for* $l = 1, 2, \ldots, m$,

(b) T_k *acts on* $1_{\mathbf{i}}M$ *as follows for* $k = 1, 2, \ldots, m-1$,

$$\frac{(pf(\varkappa_k) - p^{-1}f(\varkappa_{k+1}))(\varkappa_k - \varkappa_{k+1})}{f(\varkappa_k) - f(\varkappa_{k+1})}\sigma_k + p \qquad \text{if } i_{k+1} = i_k,$$

$$\frac{f(\varkappa_k) - f(\varkappa_{k+1})}{(p^{-1}f(\varkappa_k) - pf(\varkappa_{k+1}))(\varkappa_k - \varkappa_{k+1})}\sigma_k + \frac{(p^{-2}-1)f(\varkappa_{k+1})}{pf(\varkappa_k) - p^{-1}f(\varkappa_{k+1})} \qquad \text{if } i_{k+1} = p^2 i_k,$$

$$\frac{pi_kf(\varkappa_k) - p^{-1}i_{k+1}f(\varkappa_{k+1})}{i_kf(\varkappa_k) - i_{k+1}f(\varkappa_{k+1})}\sigma_k + \frac{(p^{-1}-p)i_kf(\varkappa_{k+1})}{i_{k+1}f(\varkappa_k) - i_kf(\varkappa_{k+1})} \qquad \text{if } i_{k+1} \neq i_k, p^2 i_k,$$

(c) T_0 *acts on* $1_{\mathbf{i}}M$ *as follows*

$$\frac{f(\varkappa_1)^2 - 1}{(q^{-1} - qf(\varkappa_1)^2)\varkappa_1}\pi_1 + \frac{(q^{-2}-1)f(\varkappa_1)^2}{q - q^{-1}f(\varkappa_1)^2} \qquad \text{if } i_1 = \pm q,$$

$$\frac{q - q^{-1}i_1^2f(\varkappa_1)^2}{1 - i_1^2f(\varkappa_1)^2}\pi_1 + \frac{q - q^{-1}}{1 - i_1^2f(\varkappa_1)^{-2}} \qquad \text{if } i_1 \neq \pm q.$$

6.6. Remark. The first case in (c) does not occur if $q = q^{-1}$ because θ has no fixed points in I. In the second case we have $i_1^2 \neq 1$ for the same reason. Note also that $(f(\varkappa) - 1)/\varkappa$ is a formal series in $\mathbf{k}[[\varkappa]]$, and that $(f(\varkappa_1) - f(\varkappa_2))/(\varkappa_1 - \varkappa_2)$ is an invertible formal series in $\mathbf{k}[[\varkappa_1 - \varkappa_2]]$. Finally, recall that $p^2 \neq 1$.

Proof : First, recall that $\pm 1 \notin I$ and that $p \neq \pm 1$. Observe also that (6.2) yields

$$i_1 = \pm q \iff \lambda_{i_1} = 1,$$

$$i_1 \neq \pm q \iff \lambda_{i_1} = 0.$$

The functor above is well defined by formulas (5.4) and (6.1). Let g be the inverse of f, i.e., $g(X)$ is the unique formal series in $\mathbf{k}[[X-1]]$ such that $gf(\varkappa) = \varkappa$. For instance, we may choose

$$f(\varkappa) = 1 + \varkappa, \quad g(X) = X - 1.$$

A quasi-inverse functor $\mathbf{H}_m\text{-}\mathbf{Mod}_I \to {}^{\theta}\mathbf{R}_m\text{-}\mathbf{Mod}_0$ such that $M \mapsto M$ is given by the following rules

(a) $1_{\mathbf{i}}M = \{m \in M;\, (i_lX_l - 1)^r m = 0,\, r \gg 0\}$,

(b) $\varkappa_l$ acts on $1_{\mathbf{i}}M$ by $g(i_lX_l)$ for $l = 1, 2, \ldots, m$,

(c) σ_k acts on $1_{\mathbf{i}}M$ as follows for $k = 1, 2, \ldots, m-1$,

$$\frac{X_k - X_{k+1}}{(pX_k - p^{-1}X_{k+1})(g(i_kX_k) - g(i_kX_{k+1}))}(T_k - p) \qquad \text{if } i_{k+1} = i_k,$$

$$\frac{g(i_{k+1}X_k) - g(i_kX_{k+1})}{pX_k - p^{-1}X_{k+1}}\Big((X_k - X_{k+1})T_k + (p - p^{-1})X_{k+1}\Big) \qquad \text{if } i_{k+1} = p^2 i_k,$$

$$T_k\frac{X_k - X_{k+1}}{p^{-1}X_k - pX_{k+1}} + (p - p^{-1})\frac{X_{k+1}}{p^{-1}X_k - pX_{k+1}} \qquad \text{if } i_{k+1} \neq i_k, p^2 i_k,$$

(*d*) π_1 acts on $1_{\mathbf{i}}M$ as follows

$$\begin{aligned} &\frac{g(i_0X_1)}{q^{-1}-qX_1^{-2}}\Big((X_1^{-2}-1)T_0+q-q^{-1}\Big) && \text{if } i_1=\pm q,\\ &T_0\frac{X_1^{-2}-1}{q^{-1}X_1^{-2}-q}+\frac{q-q^{-1}}{q^{-1}X_1^{-2}-q} && \text{if } i_1\neq\pm q.\end{aligned}$$

Note that $g(X)/(X-1)$ is a formal series in $\mathbf{k}[[X-1]]$, and that $(X_1-X_2)/(g(X_1)-g(X_2))$ is an invertible formal series in $\mathbf{k}[[X_1-X_2]]$.

□

6.7. Corollary. *There is an equivalence of categories*

$$\Psi : {}^\theta\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to \mathbf{H}_m\text{-}\mathbf{fMod}_I, \quad M\mapsto M.$$

6.8. Example. Let $m=1$. Using Example 5.9(*b*) it is easy to check that the 1-dimensional ${}^\theta\mathbf{R}_m$-modules are labelled by $\{i\in I\,;\,\lambda_i+\lambda_{\theta(i)}\neq 0\}$, and that the irreducible 2-dimensional ${}^\theta\mathbf{R}_m$-modules are labelled by $\{i\in I\,;\,\lambda_i+\lambda_{\theta(i)}=0\}/\theta$. Further we have $\lambda_i+\lambda_{\theta(i)}\neq 0$ iff $i=\pm q^{-1}$ or $\pm q$. On the other hand the 1-dimensional objects in $\mathbf{H}_m$-$\mathbf{Mod}_I$ are given by

(*a*) $X_1=i^{-1}$, $T_0=q$, $i\in I\cap\{\pm q^{-1}\}$,

(*b*) $X_1=i^{-1}$, $T_0=-q^{-1}$, $i\in I\cap\{\pm q\}$,

and the irreducible 2-dimensional objects in $\mathbf{H}_m$-$\mathbf{Mod}_I$ are given by

(*c*) $X_1=\begin{pmatrix} i & 0\\ 0 & i^{-1}\end{pmatrix}$, $T_0=\begin{pmatrix} -i^2a/b & a^2-b^2/i^2\\ -i^2/b^2 & a/b\end{pmatrix}$ with $a=q-q^{-1}$, $b=1-i^2$, and $i\neq\pm q^{-1},\pm q$.

Therefore Theorem 6.5 is obvious in this case.

6.9. Induction and restriction of $\mathbf{H}_m$-modules. For $i\in I$ we define functors

$$\begin{aligned} &E_i : \mathbf{H}_m\text{-}\mathbf{fMod}_I \to \mathbf{H}_{m-1}\text{-}\mathbf{fMod}_I,\\ &F_i : \mathbf{H}_m\text{-}\mathbf{fMod}_I \to \mathbf{H}_{m+1}\text{-}\mathbf{fMod}_I,\end{aligned} \tag{6.4}$$

where $E_iM\subset M$ is the generalized i^{-1}-eigenspace of the X_m-action, and where

$$F_iM=\mathrm{Ind}_{\mathbf{H}_m\otimes\mathbf{k}[X_{m+1}^{\pm1}]}^{\mathbf{H}_{m+1}}(M\otimes\mathbf{k}_i).$$

Here $\mathbf{k}_i$ is the 1-dimensional representation of $\mathbf{k}[X_{m+1}^{\pm1}]$ defined by $X_{m+1}\mapsto i^{-1}$.

6.10. Remark. The results in Section 6 hold true if $\mathbf{k}$ is any field of characteristic $\neq 2$. Indeed, set $f(\varkappa)=1+\varkappa$ and $g(X)=X-1$. Then we must check that the formulas for T_k, T_0 and that the formulas for σ_k, π_1 still make sense. This is straightforward for all cases, except for the first formula for π_1. Here one needs that i_1X_1+1 is invertible, which holds true if the characteristic is not 2.

7. Global bases of $\mathbf{f}$ and projective graded modules of KLR algebras

This section is a reminder on KLR algebras. Most of the results here are due to [KL]. Although we are essentially concerned by KLR algebras of type A, everything here holds true in any type.

7.1. Definition of the graded k-algebra $\mathbf{R}_m$. Fix a $\mathbb{Z} \rtimes \mathbb{Z}_2$-invariant subset $I \subset \mathbf{k}^\times$ as in Section 6.2. Let Γ be the corresponding quiver. For each integer $m \geqslant 0$ we put

$$\mathbf{R}(I)_m = \bigoplus_\nu \mathbf{R}(I)_\nu, \quad \mathbf{R}(I)_\nu = \mathbf{R}(\Gamma)_\nu,$$

where ν runs over the set of all dimension vectors in $\mathbb{N}I$ such that $|\nu| = m$. Here $\mathbf{R}(\Gamma)_\nu$ is the graded $\mathbf{k}$-algebra introduced in Section 1.3. When there is no risk of confusion we'll abbreviate $\mathbf{R}_m = \mathbf{R}(I)_m$. Let $Q_{i,j}(u,v)$ be as in (5.1). If $m > 0$ the graded $\mathbf{k}$-algebra $\mathbf{R}_m$ is generated by elements $1_\mathbf{i}$, $\varkappa_{\mathbf{i},l}$, $\sigma_{\mathbf{i},k}$ with $\mathbf{i} \in I^m$, $l = 1, 2, \dots, m$ and $k = 1, 2, \dots, m-1$ satisfying the following defining relations

(a) $1_\mathbf{i}\, 1_{\mathbf{i}'} = \delta_{\mathbf{i},\mathbf{i}'} 1_\mathbf{i}, \quad \sigma_{\mathbf{i},k} = 1_{s_k\mathbf{i}} \sigma_{\mathbf{i},k} 1_\mathbf{i}, \quad \varkappa_{\mathbf{i},l} = 1_\mathbf{i} \varkappa_{\mathbf{i},l} 1_\mathbf{i}$,

(b) $\varkappa_l \varkappa_{l'} = \varkappa_{l'} \varkappa_l$,

(c) $\sigma_k^2 1_\mathbf{i} = Q_{i_k,i_{k+1}}(\varkappa_{k+1}, \varkappa_k) 1_\mathbf{i}$,

(d) $\sigma_k \sigma_{k'} = \sigma_{k'} \sigma_k$ if $|k - k'| > 1$,

(e) $(\sigma_{k+1}\sigma_k\sigma_{k+1} - \sigma_k\sigma_{k+1}\sigma_k) 1_\mathbf{i} =$

$$= \begin{cases} \dfrac{Q_{i_k,i_{k+1}}(\varkappa_{k+1}, \varkappa_k) - Q_{i_k,i_{k+1}}(\varkappa_{k+1}, \varkappa_{k+2})}{\varkappa_k - \varkappa_{k+2}} 1_\mathbf{i} & \text{if } i_k = i_{k+2}, \\ 0 & \text{else,} \end{cases}$$

(f) $$(\sigma_k \varkappa_{k'} - \varkappa_{s_k(k')} \sigma_k) 1_\mathbf{i} = \begin{cases} -1_\mathbf{i} & \text{if } k' = k,\ i_k = i_{k+1}, \\ 1_\mathbf{i} & \text{if } k' = k+1,\ i_k = i_{k+1}, \\ 0 & \text{else.} \end{cases}$$

The grading on $\mathbf{R}_m$ is given by the following rules : $1_\mathbf{i}$ has the degree 0, $\varkappa_{\mathbf{i},l}$ has the degree 2, and $\sigma_{\mathbf{i},k}$ has the degree $-i_k \cdot i_{k+1}$. Given any element a in $1_\mathbf{i}\mathbf{R}_m 1_{\mathbf{i}'}$ we write $\varkappa_k a = \varkappa_{\mathbf{i},k} a$, $a \varkappa_k = a \varkappa_{\mathbf{i}',k}$, etc. Note that the $\mathbf{k}$-algebra $\mathbf{R}_m$ may not have 1, because the set I may be infinite. If $m = 0$ we have $\mathbf{R}_m = \mathbf{k}$ as a graded $\mathbf{k}$-algebra.

Let ω be the unique anti-involution of the graded $\mathbf{k}$-algebra $\mathbf{R}_m$ given by

$$\omega : \ 1_\mathbf{i},\ \varkappa_l,\ \sigma_k \mapsto 1_\mathbf{i},\ \varkappa_l,\ \sigma_k.$$

Note that $Q_{i,j}(u,v) = Q_{j,i}(v,u)$. Hence there is an unique involution τ of the graded $\mathbf{k}$-algebra $\mathbf{R}_m$ such that

$$\tau : \ 1_\mathbf{i},\ \varkappa_l,\ \sigma_k \mapsto 1_{w_m(\mathbf{i})},\ \varkappa_{m+1-l},\ -\sigma_{m-k},$$

where w_m is the longest element in $\mathfrak{S}_m$. Finally, we have $Q_{i,j}(u,v) = Q_{\theta(i),\theta(j)}(-u,-v)$. Hence there is an unique involution

$$\iota : \ 1_\mathbf{i},\ \varkappa_l,\ \sigma_k \mapsto 1_{\theta(\mathbf{i})},\ -\varkappa_l,\ -\sigma_k.$$

We define

$$\kappa = \iota \circ \tau = \tau \circ \iota. \tag{7.1}$$

7.2. The Grothendieck groups of $\mathbf{R}_m$. The graded $\mathbf{k}$-algebra $\mathbf{R}_m$ is finite dimensional over its center, a commutative graded $\mathbf{k}$-subalgebra. Therefore any simple object of $\mathbf{R}_m$**-mod** is finite-dimensional and there is a finite number of simple modules in $\mathbf{R}_m$**-mod**. The Abelian group $G(\mathbf{R}_m)$ is free with a basis formed by the classes of the simple objects of $\mathbf{R}_m$**-mod**, see Section 0.2 for the notation. The Abelian group $K(\mathbf{R}_m)$ is also free, with a basis formed by the classes of the indecomposable projective objects. Both Abelian groups are free $\mathcal{A}$-modules where v shifts the grading by 1. We define

$$\mathbf{K}_I = \bigoplus_{m\geqslant 0} \mathbf{K}_{I,m}, \quad \mathbf{K}_{I,m} = K(\mathbf{R}_m),$$

$$\mathbf{G}_I = \bigoplus_{m\geqslant 0} \mathbf{G}_{I,m}, \quad \mathbf{G}_{I,m} = G(\mathbf{R}_m).$$

Now, fix integers $m, m', m'' \geqslant 0$ with $m'' = m + m'$. Given sequences $\mathbf{i} \in I^m$ and $\mathbf{i}' \in I^{m'}$ we write $\mathbf{i}'' = \mathbf{i}\mathbf{i}'$. We'll abbreviate

$$\mathbf{R}_{m,m'} = \mathbf{R}_m \otimes \mathbf{R}_{m'}.$$

There is an unique inclusion of graded $\mathbf{k}$-algebras

$$\begin{aligned}
&\phi : \mathbf{R}_{m,m'} \to \mathbf{R}_{m''},\\
&1_{\mathbf{i}} \otimes 1_{\mathbf{i}'} \mapsto 1_{\mathbf{i}''},\\
&\varkappa_{\mathbf{i},l} \otimes 1_{\mathbf{i}'} \mapsto \varkappa_{\mathbf{i}'',l},\\
&1_{\mathbf{i}} \otimes \varkappa_{\mathbf{i}',l} \mapsto \varkappa_{\mathbf{i}'',m+l},\\
&\sigma_{\mathbf{i},k} \otimes 1_{\mathbf{i}'} \mapsto \sigma_{\mathbf{i}'',k},\\
&1_{\mathbf{i}} \otimes \sigma_{\mathbf{i}',k} \mapsto \sigma_{\mathbf{i}'',m+k}.
\end{aligned} \tag{7.2}$$

This yields a triple of adjoint functors $(\phi_!, \phi^*, \phi_*)$ where

$$\phi^* : \mathbf{R}_{m''}\text{-}\mathbf{mod} \to \mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_{m'}\text{-}\mathbf{mod}$$

is the restriction and $\phi_!$, ϕ_* are given by

$$\phi_! : \begin{cases} \mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_{m'}\text{-}\mathbf{mod} \to \mathbf{R}_{m''}\text{-}\mathbf{mod},\\ (M, M') \mapsto \mathbf{R}_{m''} \otimes_{\mathbf{R}_{m,m'}} (M \otimes M'), \end{cases}$$

$$\phi_* : \begin{cases} \mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_{m'}\text{-}\mathbf{mod} \to \mathbf{R}_{m''}\text{-}\mathbf{mod},\\ (M, M') \mapsto \hom_{\mathbf{R}_{m,m'}}(\mathbf{R}_{m''}, M \otimes M'). \end{cases}$$

First, note that the functors $\phi_!$, ϕ^*, ϕ_* commute with the shift of the grading. Next, the functor ϕ^* is exact and it takes finite dimensional graded modules to finite dimensional ones. By [KL, prop. 2.16] the right graded $\mathbf{R}_{m,m'}$-module $\mathbf{R}_{m''}$ is free of finite rank. Thus $\phi_!$ is exact and it takes finite dimensional graded modules to finite dimensional ones. For the same reason the left graded $\mathbf{R}_{m,m'}$-module $\mathbf{R}_{m''}$ is free of finite rank. Thus ϕ_* is exact and it takes finite dimensional graded modules to finite dimensional ones. Further $\phi_!$ and ϕ^* take projective graded modules to projective ones, because they are left adjoint to the exact functors ϕ^*, ϕ_* respectively. To summarize, the functors $\phi_!$, ϕ^*, ϕ_* are exact and take finite dimensional graded modules to finite dimensional ones, and the functors $\phi_!$, ϕ^* take projective graded modules to projective ones. Taking the sum over all m, m' we get an $\mathcal{A}$-bilinear map

$$\phi_! : \mathbf{K}_I \times \mathbf{K}_I \to \mathbf{K}_I.$$

In the same way we define also an $\mathcal{A}$-linear map

$$\phi^* : \mathbf{K}_I \to \mathbf{K}_I \otimes_{\mathcal{A}} \mathbf{K}_I.$$

From now on, to unburden the notation we may abbreviate $\mathbf{R} = \mathbf{R}_m$, hoping it will not create any confusion. Recall the anti-automorphism ω from the previous section. Consider the duality

$$\mathbf{R}\text{-}\mathbf{proj} \to \mathbf{R}\text{-}\mathbf{proj}, \quad P \mapsto P^\sharp = \hom_{\mathbf{R}}(P, \mathbf{R}),$$

with the action and the grading given by

$$(xf)(p) = f(p)\omega(x), \quad (P^\sharp)_d = \mathrm{Hom}_{\mathbf{R}}(P[-d], \mathbf{R}).$$

We'll say that P is $\sharp$-selfdual if $P^\sharp = P$. The duality on $\mathbf{R}$-$\mathbf{proj}$ yields an $\mathcal{A}$-antilinear map

$$\mathbf{K}_I \to \mathbf{K}_I, \quad P \mapsto P^\sharp.$$

Set $\mathcal{B} = \mathbb{Z}((v))$. The $\mathcal{A}$-module $\mathbf{K}_I$ is equipped with a symmetric $\mathcal{A}$-bilinear form

$$\mathbf{K}_I \times \mathbf{K}_I \to \mathcal{B}, \quad (P : Q) = \mathrm{gdim}(P^\omega \otimes_{\mathbf{R}} Q).$$

Here P^ω is the right graded $\mathbf{R}$-module associated with P and the anti-automorphism ω. Finally, we equip $\mathbf{K}_I \otimes_{\mathcal{A}} \mathbf{K}_I$ with the algebra structure such that

$$(P \otimes Q, P' \otimes Q') \mapsto v^{-\mu\cdot\nu'} \phi_!(P, P') \otimes \phi_!(Q, Q'),$$

and with the $\mathcal{A}$-antilinear map such that

$$P \otimes Q \mapsto (P \otimes Q)^\sharp = P^\sharp \otimes Q^\sharp.$$

The following is proved in [KL].

7.3. Proposition. *The map $\phi_!$ turns $\mathbf{K}_I$ into an associative $\mathcal{A}$-algebra with 1, and it commutes with the duality $\sharp$. The map ϕ^* is an algebra homomorphism which turns $\mathbf{K}_I$ into a coassociative $\mathcal{A}$-coalgebra.*

The Cartan pairing is the perfect $\mathcal{A}$-bilinear form

$$\mathbf{K}_I \times \mathbf{G}_I \to \mathcal{A}, \quad \langle P : M \rangle = \mathrm{gdim}\,\hom_{\mathbf{R}}(P, M).$$

Consider the duality

$$\mathbf{R}\text{-}\mathbf{fmod} \to \mathbf{R}\text{-}\mathbf{fmod}, \quad M \mapsto M^\flat = \hom(M, \mathbf{k}),$$

where $\mathbf{k}$ is considered as a graded $\mathbf{k}$-space homogeneous of degree 0. The action and the grading are given by

$$(xf)(m) = f(\omega(x)m), \quad (M^\flat)_d = \mathrm{Hom}(M_{-d}, \mathbf{k}).$$

We'll say that M is $\flat$-selfdual if $M^\flat = M$.

Finally, let $\mathcal{B}I^m$ be the free $\mathcal{B}$-module with basis I^m. The *character* of a finitely generated graded $\mathbf{R}_m$-module M is given by

$$\mathrm{ch}(M) = \sum_{\mathbf{i}} \mathrm{gdim}(1_{\mathbf{i}}M)\,\mathbf{i} \in \mathcal{B}I^m.$$

7.4. The projective graded $\mathbf{R}_m$-module $\mathbf{R}_\mathbf{y}$. Fix $\nu \in \mathbb{N}I$ with $|\nu| = m$. For $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ in Y^ν we define an object $\mathbf{R}_\mathbf{y}$ in $\mathbf{R}_m$-**proj** as follows.

- If $\mathbf{i} = i^m$, $i \in I$, and $\mathbf{a} = m$ then we set $\mathbf{R}_\mathbf{y} = \mathbf{F}_\nu[\ell_m]$. As a left graded $\mathbf{R}_\nu$-modules we have a canonical isomorphism $\mathbf{R}_\nu = \bigoplus_{w\in\mathfrak{S}_m} \mathbf{R}_\mathbf{y}[2\ell(w) - \ell_m]$. We choose once for all an idempotent $1_\mathbf{y}$ in $\mathbf{R}_m$ such that $\mathbf{R}_\mathbf{y} = (\mathbf{R}_m 1_\mathbf{y})[\ell_m]$.
- If $\mathbf{i} = (i_1, \dots, i_k)$ and $\mathbf{a} = (a_1, \dots a_k)$ we define the idempotent $1_\mathbf{y}$ as the image of the element $\bigotimes_{l=1}^k 1_{(i_l)^{a_l}, a_l}$ by the inclusion of graded $\mathbf{k}$-algebras $\bigotimes_{l=1}^k \mathbf{R}_{a_l i_l} \subset \mathbf{R}_\nu$ in (7.2). Then we set $\mathbf{R}_\mathbf{y} = (\mathbf{R}_\nu 1_\mathbf{y})[\ell_\mathbf{a}]$.

The graded module $\mathbf{R}_\mathbf{y}$ satisfies the following properties.

- Let $\mathbf{i}' \in I^\nu$ be the sequence obtained by expanding the pair $\mathbf{y} = (\mathbf{i}, \mathbf{a})$. We have the following formula in $\mathbf{R}_m$-**proj**

$$\mathbf{R}_{\mathbf{i}'} = \mathbf{R}1_{\mathbf{i}'} = \bigoplus_{w\in\mathfrak{S}_\mathbf{a}} \mathbf{R}_\mathbf{y}[2\ell(w) - \ell_\mathbf{a}] =: \langle\mathbf{a}\rangle!\,\mathbf{R}_\mathbf{y}.$$

  As a consequence, since the graded module $\mathbf{R}_{\mathbf{i}'}$ is $\sharp$-selfdual we get

$$\mathbf{R}_\mathbf{y}[d]^\sharp = \mathbf{R}_\mathbf{y}[-d], \quad \forall d \in \mathbb{Z}.$$

- Given $\mathbf{y} = (\mathbf{i}, \mathbf{a}) \in Y_\nu$ and $\mathbf{y}' = (\mathbf{i}', \mathbf{a}') \in Y_{\nu'}$ we set $\mathbf{y}\mathbf{y}' = (\mathbf{i}\mathbf{i}', \mathbf{a}\mathbf{a}')$. We have an isomorphism of graded $\mathbf{R}_{\nu''}$-modules $\phi_!(\mathbf{R}_\mathbf{y} \otimes \mathbf{R}_{\mathbf{y}'}) = \mathbf{R}_{\mathbf{y}''}$.

7.5. Examples. For $\mathbf{i} \in I^\nu$, $|\nu| = m$ and $\mathbf{y} \in Y^\nu$, we define $\mathbf{L}_\mathbf{i} = \mathrm{top}(\mathbf{R}_\mathbf{i})$ and $\mathbf{L}_\mathbf{y} = \mathrm{top}(\mathbf{R}_\mathbf{y})$. Observe that $\mathbf{L}_\mathbf{i}$ is not a simple graded $\mathbf{R}_m$-module in general.

(a) The graded $\mathbf{k}$-algebra $\mathbf{R}_1$ is generated by elements 1_i, $\varkappa_i$, $i \in I$, satisfying the defining relations $1_i\,1_{i'} = \delta_{i,i'}1_i$ and $\varkappa_i = 1_i\varkappa_i 1_i$. Note that $\mathbf{R}_i = 1_i\mathbf{R}_1 = \mathbf{R}_1 1_i$ is a graded subalgebra of $\mathbf{R}_1$, and that $\mathbf{L}_i = \mathbf{R}_i/(\varkappa_i) = \mathbf{k}$. Further, we have $\mathrm{ch}(\mathbf{R}_i) \in (1 - v^2)^{-1} i$ and $\mathrm{ch}(\mathbf{L}_i) = i$, where the symbol $(1 - v^2)^{-1}$ denotes the infinite sum $\sum_{r\geqslant 0} v^{2r}$.

(b) Set $\nu = mi$ and $\mathbf{y} = (i, m)$. We'll abbreviate $\mathbf{L}_{mi} = \mathbf{L}_{i,m} = \mathbf{L}_\mathbf{y}$. It is a simple graded $\mathbf{R}_m$-module. We have

$$\mathrm{ch}(\mathbf{R}_{mi}) \in v^{-\ell_m}\mathbb{Z}[[v^2]]\, i^m, \quad \mathrm{ch}(\mathbf{L}_{mi}) = \langle m\rangle!\, i^m.$$

The graded $\mathbf{R}_{m-1} \otimes \mathbf{R}_1$-module $\mathbf{L}_{mi}$ has a filtration by graded submodules whose associated graded is isomorphic to $[m]\,\mathbf{L}_{(m-1)i} \otimes \mathbf{L}_i$. The socle of the graded $\mathbf{R}_{m-n} \otimes \mathbf{R}_n$-module $\mathbf{L}_{mi}$ is equal to $\mathbf{L}_{(m-n)i} \otimes \mathbf{L}_{ni}$ for each $m \geqslant n \geqslant 0$. See [KL, ex. 2.2, prop. 3.11] for details.

7.6. Categorification of the global bases of $\mathbf{f}$. Set $\mathcal{K}=\mathbb{Q}(v)$. Let $\mathbf{f}$ be the $\mathcal{K}$-algebra generated by elements θ_i, $i\in I$, with the defining relations

$$\sum_{a+b=1-i\cdot j}(-1)^a\theta_i^{(a)}\theta_j\theta_i^{(b)}=0,\quad i\neq j,\quad \theta_i^{(a)}=\theta_i^a/\langle a\rangle!,\quad a\geqslant 0. \tag{7.3}$$

We have a weight decomposition $\mathbf{f}=\bigoplus_{\nu\in\mathbb{N}I}\mathbf{f}_\nu$. Let ${}_{\mathcal{A}}\mathbf{f}$ be the $\mathcal{A}$-submodule of $\mathbf{f}$ generated by all products of the elements $\theta_i^{(a)}$ with $a\in\mathbb{Z}_{\geqslant 0}$ and $i\in I$. We set

$${}_{\mathcal{A}}\mathbf{f}_\nu={}_{\mathcal{A}}\mathbf{f}\cap\mathbf{f}_\nu.$$

The element θ_i lies in ${}_{\mathcal{A}}\mathbf{f}_i$ for each $i\in I$. For each pair $\mathbf{y}=(\mathbf{i},\mathbf{a})$ in Y^ν with $\mathbf{i}=(i_1,\dots,i_k)$, $\mathbf{a}=(a_1,\dots a_k)$ we write

$$\theta_{\mathbf{y}}=\theta_{i_1}^{(a_1)}\theta_{i_2}^{(a_2)}\cdots\theta_{i_k}^{(a_k)}.$$

We equip ${}_{\mathcal{A}}\mathbf{f}$ with the unique $\mathcal{A}$-antilinear involution such that $\bar\theta_i=\theta_i$ for each $i\in I$. We equip the tensor square of $\mathbf{f}$ with the $\mathcal{K}$-algebra structure such that

$$(x\otimes y)(x'\otimes y')=v^{-\mu\cdot\nu'}xx'\otimes yy',\quad x\in\mathbf{f}_\nu,\ x'\in\mathbf{f}_{\nu'},\ y\in\mathbf{f}_\mu,\ y'\in\mathbf{f}_{\mu'}.$$

Consider the $\mathcal{K}$-algebra homomorphism such that

$$r:\mathbf{f}\to\mathbf{f}\otimes\mathbf{f},\quad \theta_i\mapsto\theta_i\otimes 1+1\otimes\theta_i.$$

The $\mathcal{K}$-algebra $\mathbf{f}$ comes equipped with a bilinear form $(\bullet:\bullet)$ which is uniquely determined by the following conditions

- $(1:1)=1$,
- $(\theta_i:\theta_j)=\delta_{i,j}(1-v^2)^{-1}$ for all $i,j\in I$,
- $(x:yy')=(r(x):y\otimes y')$ for all x,y,y',
- $(xx':y)=(x\otimes x':r(y))$ for all x,x',y.

This bilinear form is symmetric and non-degenerate. Let $\theta^i\in{}_{\mathcal{A}}\mathbf{f}_i^*$ be the element dual to θ_i. We may regard

$${}_{\mathcal{A}}\mathbf{f}^*=\bigoplus_\nu{}_{\mathcal{A}}\mathbf{f}_\nu^*,\quad {}_{\mathcal{A}}\mathbf{f}_\nu^*=\mathrm{Hom}_{\mathcal{A}}({}_{\mathcal{A}}\mathbf{f}_\nu,\mathcal{A}),$$

as an $\mathcal{A}$-submodule of $\mathbf{f}$ via the bilinear form $(\bullet:\bullet)$. Let $\mathbf{G}^{\mathrm{low}}$ be the *canonical basis (=the lower global basis)* of $\mathbf{f}$. It is indeed a $\mathcal{A}$-basis of ${}_{\mathcal{A}}\mathbf{f}$. The *upper global basis of* $\mathbf{f}$ is the $\mathcal{K}$-basis $\mathbf{G}^{\mathrm{up}}$ which is dual to $\mathbf{G}^{\mathrm{low}}$ with respect to the inner product $(\bullet:\bullet)$. We may regard $\mathbf{G}^{\mathrm{up}}$ as a $\mathcal{K}$-basis of $\mathbf{f}^*$. Let $B(\infty)$ be the set of isomorphism classes of irreducible (non graded) $\mathbf{R}$-modules such that the elements $\varkappa_1,\varkappa_2,\dots,\varkappa_m$ act nilpotently.

7.7. Theorem. *(a) There is an unique $\mathcal{A}$-algebra isomorphism $\gamma : {}_{\mathcal{A}}\mathbf{f} \to \mathbf{K}_I$ which intertwines r and ϕ^*, and such that $\gamma(\theta_{\mathbf{y}}) = \mathbf{R}_{\mathbf{y}}$ for each $\mathbf{y}$.*

(b) We have $\mathbf{G}^{\mathrm{low}} = \{G^{\mathrm{low}}(b);\, b \in B(\infty)\}$, where $\gamma(G^{\mathrm{low}}(b))$ is the unique $\sharp$-selfdual indecomposable projective graded module whose top is isomorphic to b. The map γ takes the bilinear form $(\bullet : \bullet)$ and the involution $\bar{\bullet}$ on ${}_{\mathcal{A}}\mathbf{f}$ to the bilinear form $(\bullet : \bullet)$ and the involution $\bullet^\sharp$ on $\mathbf{K}_I$.

(c) The transpose map $\mathbf{G}_I \to {}_{\mathcal{A}}\mathbf{f}^*$ takes the $\mathcal{A}$-basis of $\mathbf{G}_I$ of the $\flat$-selfdual simple objects to $\mathbf{G}^{\mathrm{up}}$. We have ${}^t\gamma(\mathbf{L}_i) = \theta^i$ for all $i \in I$, and $\mathbf{G}^{\mathrm{up}} = \{G^{\mathrm{up}}(b);\, b \in B(\infty)\}$ with $G^{\mathrm{up}}(b) = {}^t\gamma\,\mathrm{top}\,\gamma\, G^{\mathrm{low}}(b)$.

Proof : Claim (a), and the second part of (b) are due to [KL, prop. 3.4]. The first part of (b) is due to [VV] (the same result has also been anounced by R. Rouquier). Part (c) follows form (b). For instance, the last claim in (c) is as proved as follows. Let $\langle \bullet : \bullet \rangle$ denote both the Cartan pairing and the canonical pairing

$$ {}_{\mathcal{A}}\mathbf{f} \times {}_{\mathcal{A}}\mathbf{f}^* \to \mathcal{A}. $$

Then we have

$$ \langle \mathbf{b}' : {}^t\gamma\,\mathrm{top}\,\gamma(\mathbf{b}) \rangle = \langle \gamma(\mathbf{b}') : \mathrm{top}\,\gamma(\mathbf{b}) \rangle = \delta_{\mathbf{b},\mathbf{b}'}. $$

$\square$

8. Global bases of ${}^\theta\mathbf{V}(\lambda)$ and projective graded ${}^\theta\mathbf{R}$-modules

Given an integer $m \geqslant 0$ we consider the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}_m$ introduced in Sections 5.1, 6.4.

8.1. The Grothendieck groups of ${}^\theta\mathbf{R}_m$. The graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}_m$ is free of finite type over its center by Proposition 5.7(b). Therefore any simple object of ${}^\theta\mathbf{R}_m$-**mod** is finite-dimensional and there is a finite number of isomorphism classes of simple modules in ${}^\theta\mathbf{R}_m$-**mod**. Further, the Abelian group $G({}^\theta\mathbf{R}_m)$ is free with a basis formed by the classes of the simple objects of ${}^\theta\mathbf{R}_m$-**mod**. For each ν the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}_\nu$ has a graded dimension which lies in $v^d\mathbb{N}[[v]]$ for some integer d. Therefore the Abelian group $K({}^\theta\mathbf{R}_m)$ is free with a basis formed by the classes of the indecomposable projective objects. Both $G({}^\theta\mathbf{R})$ and $K({}^\theta\mathbf{R})$ are free $\mathcal{A}$-modules where v shifts the grading by 1. We consider the following $\mathcal{A}$-modules

$$ {}^\theta\mathbf{K}_I = \bigoplus_{m \geqslant 0} {}^\theta\mathbf{K}_{I,m}, \quad {}^\theta\mathbf{K}_{I,m} = K({}^\theta\mathbf{R}_m), $$

$$ {}^\theta\mathbf{G}_I = \bigoplus_{m \geqslant 0} {}^\theta\mathbf{G}_{I,m}, \quad {}^\theta\mathbf{G}_{I,m} = G({}^\theta\mathbf{R}_m). $$

From now on, to unburden the notation we may abbreviate ${}^\theta\mathbf{R} = {}^\theta\mathbf{R}_m$, hoping it will not create any confusion. For any M, N in ${}^\theta\mathbf{R}$-**mod** we set

$$ (M : N) = \mathrm{gdim}(M^\omega \otimes_{{}^\theta\mathbf{R}} N), \quad \langle M : N \rangle = \mathrm{gdim}\,\mathrm{hom}_{{}^\theta\mathbf{R}}(M, N). \tag{8.1} $$

Here M^ω is the right graded ${}^\theta\mathbf{R}$-module associated with M and the anti-automorphism ω introduced in Section 5.1. The Cartan pairing is the perfect $\mathcal{A}$-bilinear form

$$ {}^\theta\mathbf{K}_I \times {}^\theta\mathbf{G}_I \to \mathcal{A}, \quad (P, M) \mapsto \langle P : M \rangle, $$

see (8.1). First, we concentrate on the $\mathcal{A}$-module ${}^\theta\mathbf{G}_I$. Consider the duality

$$ {}^\theta\mathbf{R}\text{-}\mathbf{fmod} \to {}^\theta\mathbf{R}\text{-}\mathbf{fmod}, \quad M \mapsto M^\flat = \hom(M, \mathbf{k}), $$

with the action and the grading given by

$$ (xf)(m) = f(\omega(x)m), \quad (M^\flat)_d = \mathrm{Hom}(M_{-d}, \mathbf{k}). $$

We'll say that M is $\flat$-selfdual if $M^\flat = M$. The functor $\flat$ yields an $\mathcal{A}$-antilinear map

$$ {}^\theta\mathbf{G}_I \to {}^\theta\mathbf{G}_I, \quad M \mapsto M^\flat. $$

We can now define the upper global basis of ${}^\theta\mathbf{G}_I$ as follows. The proof is given in Section 8.26.

8.2. Proposition/Definition. *Let ${}^\theta B(\lambda)$ be the set of isomorphism classes of simple objects in ${}^\theta\mathbf{R}\text{-}\mathbf{fMod}_0$. For each b in ${}^\theta B(\lambda)$ there is a unique $\flat$-selfdual irreducible graded ${}^\theta\mathbf{R}$-module ${}^\theta G^{\mathrm{up}}(b)$ which is isomorphic to b as a (non graded) ${}^\theta\mathbf{R}$-module. We define a $\mathcal{A}$-basis ${}^\theta\mathbf{G}^{\mathrm{up}}(\lambda)$ of ${}^\theta\mathbf{G}_I$ by setting*

$$ {}^\theta\mathbf{G}^{\mathrm{up}}(\lambda) = \{{}^\theta G^{\mathrm{up}}(b);\, b \in {}^\theta B(\lambda)\}, \quad {}^\theta G^{\mathrm{up}}(0) = 0. $$

Now, we concentrate on the $\mathcal{A}$-module ${}^\theta\mathbf{K}_I$. We equip ${}^\theta\mathbf{K}_I$ with the symmetric $\mathcal{A}$-bilinear form

$$ {}^\theta\mathbf{K}_I \times {}^\theta\mathbf{K}_I \to \mathcal{B}, \quad (P, Q) \mapsto (P : Q), \tag{8.2} $$

see (8.1). Consider the duality

$$ {}^\theta\mathbf{R}\text{-}\mathbf{proj} \to {}^\theta\mathbf{R}\text{-}\mathbf{proj}, \quad P \mapsto P^\sharp = \hom_{{}^\theta\mathbf{R}}(P, {}^\theta\mathbf{R}), $$

with the action and the grading given by

$$ (xf)(p) = f(p)\omega(x), \quad (P^\sharp)_d = \mathrm{Hom}_{{}^\theta\mathbf{R}}(P[-d], {}^\theta\mathbf{R}). $$

This duality functor yields an $\mathcal{A}$-antilinear map

$$ {}^\theta\mathbf{K}_I \to {}^\theta\mathbf{K}_I, \quad P \mapsto P^\sharp. $$

Let $\mathcal{K} \to \mathcal{K}$, $f \mapsto \bar{f}$ be the unique involution such that $\bar{v} = v^{-1}$.

8.3. Definition. *For each b in ${}^\theta B(\lambda)$ let ${}^\theta G^{\mathrm{low}}(b)$ be the unique indecomposable graded module in ${}^\theta\mathbf{R}\text{-}\mathbf{proj}$ whose top is isomorphic to ${}^\theta G^{\mathrm{up}}(b)$. We set ${}^\theta G^{\mathrm{low}}(0) = 0$ and ${}^\theta\mathbf{G}^{\mathrm{low}}(\lambda) = \{{}^\theta G^{\mathrm{low}}(b);\, b \in {}^\theta B(\lambda)\}$, a $\mathcal{A}$-basis of ${}^\theta\mathbf{K}_I$.*

8.4. Proposition. *(a) We have* $\langle {}^\theta G^{\mathrm{low}}(b) : {}^\theta G^{\mathrm{up}}(b')\rangle = \delta_{b,b'}$ *for each* b, b' *in* ${}^\theta B(\lambda)$.

(b) We have $\langle P^\sharp : M\rangle = \overline{\langle P : M^\flat\rangle}$ *for each* P, M.

(c) The graded ${}^\theta\mathbf{R}$*-module* ${}^\theta G^{\mathrm{low}}(b)$ *is* $\sharp$*-selfdual for each* b *in* ${}^\theta B(\lambda)$.

Proof : Part (a) is obvious because we have

$$\langle {}^\theta G^{\mathrm{low}}(b) : {}^\theta G^{\mathrm{up}}(b')\rangle = \mathrm{gdim}\,\hom_{{}^\theta\mathbf{R}}({}^\theta G^{\mathrm{low}}(b), \mathrm{top}\,{}^\theta G^{\mathrm{low}}(b')) = \delta_{b,b'}.$$

Part (c) is a consequence of (b). Finally (b) is proved as follows

$$\begin{aligned}\langle P^\sharp : M\rangle &= \mathrm{gdim}\,\hom_{{}^\theta\mathbf{R}}(\hom_{{}^\theta\mathbf{R}}(P, {}^\theta\mathbf{R}), M),\\ &= \mathrm{gdim}\,(P^\omega \otimes_{{}^\theta\mathbf{R}} M),\\ &= \overline{\mathrm{gdim}\,\hom(P^\omega \otimes_{{}^\theta\mathbf{R}} M, \mathbf{k})},\\ &= \overline{\mathrm{gdim}\,\hom_{{}^\theta\mathbf{R}}(P, M^\flat)},\\ &= \overline{\langle P : M^\flat\rangle}.\end{aligned}$$

Here, the second equality holds because P is a projective graded module and the fourth one is adjointness of $\otimes$ and Hom, see e.g., [CuR,(2.19)].

$\square$

8.5. Example. Set $\nu = i + \theta(i)$ and $\mathbf{i} = i\theta(i)$. Set ${}^\theta\mathbf{R}_{\mathbf{i}} = {}^\theta\mathbf{R}1_{\mathbf{i}}$ and ${}^\theta\mathbf{L}_{\mathbf{i}} = \mathrm{top}({}^\theta\mathbf{R}_{\mathbf{i}})$. Recall the description of ${}^\theta\mathbf{R}_1$ given in Example 5.9. Recall also that ${}^\theta\mathbf{R}_0 = \mathbf{k}$. The global bases are given by the following. First, the weight zero parts are given by ${}^\theta\mathbf{G}_0^{\mathrm{low}}(\lambda) = {}^\theta\mathbf{G}_0^{\mathrm{up}}(\lambda) = \{\mathbf{k}\}$. Next, let us consider the weight ν parts.

- If $\lambda_i + \lambda_{\theta(i)} \neq 0$ then ${}^\theta\mathbf{G}_\nu^{\mathrm{low}}(\lambda) = \{{}^\theta\mathbf{R}_{\mathbf{i}},\, {}^\theta\mathbf{R}_{\theta(\mathbf{i})}\}$ and ${}^\theta\mathbf{G}_\nu^{\mathrm{up}}(\lambda) = \{{}^\theta\mathbf{L}_{\mathbf{i}},\, {}^\theta\mathbf{L}_{\theta(\mathbf{i})}\}$.
- If $\lambda_i + \lambda_{\theta(i)} = 0$ then ${}^\theta\mathbf{G}_\nu^{\mathrm{low}}(\lambda) = \{{}^\theta\mathbf{R}_{\mathbf{i}}\}$, ${}^\theta\mathbf{G}_\nu^{\mathrm{up}}(\lambda) = \{{}^\theta\mathbf{L}_{\mathbf{i}}\}$, ${}^\theta\mathbf{R}_{\mathbf{i}} = {}^\theta\mathbf{R}_{\theta(\mathbf{i})}$, and ${}^\theta\mathbf{L}_{\mathbf{i}} = {}^\theta\mathbf{L}_{\theta(\mathbf{i})}$.

8.6. Definition of the operators e_i and f_i. First, let us introduce the following notation for a future use. Given integers $m, m', n, n' \geqslant 0$ such that

$$m + m' = n + n' = m'',$$

let $D_{m,m'}$ be the set of minimal representative in $W_{m''}$ of the cosets in

$$W_{m,m'} \setminus W_{m''}, \quad W_{m,m'} = W_m \times \mathfrak{S}_{m'}.$$

Recall that $W_{m''}$ is regarded as a Weyl group of type $B_{m''}$, see Section 5.4. Write

$$D_{m,m';n,n'} = D_{m,m'} \cap (D_{n,n'})^{-1}.$$

For each element w of $D_{m,m';n,n'}$ we set

$$W(w) = W_{m,m'} \cap w(W_{n,n'})w^{-1}.$$

We abbreviate

$$ {}^\theta\mathbf{R}_{m,m'} = {}^\theta\mathbf{R}_m \otimes \mathbf{R}_{m'}. $$

For any integers $m'_1, m'_2, \dots, m'_r \geqslant 0$ we define the graded $\mathbf{k}$-algebra

$$ {}^\theta\mathbf{R}_{m,m'_1,m'_2,\dots,m'_r} $$

in the same way. There is an unique inclusion of graded $\mathbf{k}$-algebras

$$
\begin{aligned}
&\psi : {}^\theta\mathbf{R}_{m,m'} \to {}^\theta\mathbf{R}_{m''},\\
&1_{\mathbf{i}} \otimes 1_{\mathbf{i}'} \mapsto 1_{\mathbf{i}''},\\
&1_{\mathbf{i}} \otimes \varkappa_{\mathbf{i}',l} \mapsto \varkappa_{\mathbf{i}'',m+l},\\
&1_{\mathbf{i}} \otimes \sigma_{\mathbf{i}',k} \mapsto \sigma_{\mathbf{i}'',m+k},\\
&\varkappa_{\mathbf{i},l} \otimes 1_{\mathbf{i}'} \mapsto \varkappa_{\mathbf{i}'',l},\\
&\pi_{\mathbf{i},1} \otimes 1_{\mathbf{i}'} \mapsto \pi_{\mathbf{i}'',1},\\
&\sigma_{\mathbf{i},k} \otimes 1_{\mathbf{i}'} \mapsto \sigma_{\mathbf{i}'',k},
\end{aligned}
\tag{8.3}
$$

where $\mathbf{i} \in {}^\theta I^m$, $\mathbf{i}' \in I^{m'}$, and $\mathbf{i}'' = \theta(\mathbf{i}')\mathbf{i}\mathbf{i}'$ is a sequence in ${}^\theta I^{m''}$.

8.7. Lemma. *The graded ${}^\theta\mathbf{R}_{m,m'}$-module ${}^\theta\mathbf{R}_{m''}$ is free of rank $2^{m'}\binom{m''}{m}$.*

Proof : Set $\nu'' = \theta(\nu') + \nu + \nu'$, where ν, ν' are vector dimensions in ${}^\theta\mathbb{N}I$, $\mathbb{N}I$ respectively, such that $|\nu| = 2m$ and $|\nu'| = m'$. For each w in $D_{m,m'}$ we have the element $\sigma_{\dot w}$ in ${}^\theta\mathbf{R}_{m''}$ defined in (5.2). Using filtered/graded arguments it is easy to see that

$$ {}^\theta\mathbf{R}_{m''} = \bigoplus_{w \in D_{m,m'}} {}^\theta\mathbf{R}_{m,m'}\sigma_{\dot w}. $$

□

Now, we consider the triple of adjoint functors $(\psi_!, \psi^*, \psi_*)$ where

$$ \psi^* \,:\, {}^\theta\mathbf{R}_{m''}\text{-}\mathbf{mod} \to {}^\theta\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_{m'}\text{-}\mathbf{mod} $$

is the restriction and $\psi_!$, ψ_* are given by

$$ \psi_! \,:\, \begin{cases} {}^\theta\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_{m'}\text{-}\mathbf{mod} \to {}^\theta\mathbf{R}_{m''}\text{-}\mathbf{mod},\\ (M, M') \mapsto {}^\theta\mathbf{R}_{m''} \otimes_{{}^\theta\mathbf{R}_{m,m'}} (M \otimes M'), \end{cases} $$

$$ \psi_* \,:\, \begin{cases} {}^\theta\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_{m'}\text{-}\mathbf{mod} \to {}^\theta\mathbf{R}_{m''}\text{-}\mathbf{mod},\\ (M, M') \mapsto \hom_{{}^\theta\mathbf{R}_{m,m'}}({}^\theta\mathbf{R}_{m''}, M \otimes M'). \end{cases} $$

The same discussion as for the triple $(\phi_!, \phi^*, \phi_*)$ implies that $\psi_!$, ψ^*, ψ_* are exact, they commute with the shift of the grading, and they take finite dimensional modules to finite dimensional ones, while the functors $\psi_!$, ψ^* take projective modules to projective ones. Thus the functor $\psi_!$ yields $\mathcal{A}$-bilinear maps

$$ \mathbf{K}_I^\theta \times \mathbf{K}_I \to {}^\theta\mathbf{K}_I, \quad {}^\theta\mathbf{G}_I \times \mathbf{G}_I \to {}^\theta\mathbf{G}_I, $$

while ψ^* yields maps in the inverse way. For a graded ${}^\theta\mathbf{R}_m$-module M we write

$$
\begin{aligned}
f_i(M) &= {}^\theta\mathbf{R}_{m+1}1_{m,i} \otimes_{{}^\theta\mathbf{R}_m} M,\\
e_i(M) &= {}^\theta\mathbf{R}_{m-1} \otimes_{{}^\theta\mathbf{R}_{m-1,1}} 1_{m-1,i}M.
\end{aligned} \tag{8.4}
$$

Let us explain these formulas. The symbols $1_{m,i}$ and $1_{m-1,i}$ are as in (6.3). Note that $f_i(M)$ is a graded ${}^\theta\mathbf{R}_{m+1}$-module, while $e_i(M)$ is a graded ${}^\theta\mathbf{R}_{m-1}$-module. The tensor product in the definition of $e_i(M)$ is relative to the graded $\mathbf{k}$-algebra homomorphism

$$
{}^\theta\mathbf{R}_{m-1,1} = {}^\theta\mathbf{R}_{m-1} \otimes \mathbf{R}_1 \to {}^\theta\mathbf{R}_{m-1} \otimes \mathbf{R}_i \to {}^\theta\mathbf{R}_{m-1} \otimes (\mathbf{R}_i/(\varkappa_i)) = {}^\theta\mathbf{R}_{m-1}.
$$

In other words, let $e'_i(M)$ is the graded ${}^\theta\mathbf{R}_{m-1}$-module obtained by taking the direct summand $1_{m-1,i}M$ and restricting it to ${}^\theta\mathbf{R}_{m-1}$. Observe that if M is finitely generated then $e'_i(M)$ may not lie in ${}^\theta\mathbf{R}_{m-1}$-**mod**. To remedy this, since $e'_i(M)$ affords a ${}^\theta\mathbf{R}_{m-1} \otimes \mathbf{R}_i$-action we consider the graded ${}^\theta\mathbf{R}_{m-1}$-module

$$
e_i(M) = e'_i(M)/\varkappa_i e'_i(M).
$$

8.8. Definition. *The functors e_i, f_i preserve the category ${}^\theta\mathbf{R}$-**proj**, yielding $\mathcal{A}$-linear operators on ${}^\theta\mathbf{K}_I$. Let e_i, f_i denote also the $\mathcal{A}$-linear operators on ${}^\theta\mathbf{G}_I$ which are the transpose of f_i, e_i with respect to the Cartan pairing.*

Note that the symbols $e_i(M)$, $f_i(M)$ have different meaning if M is viewed as an element of ${}^\theta\mathbf{K}_I$ or if M is viewed as an element of ${}^\theta\mathbf{G}_I$. In the first case they are given by (8.4), in the second one by the formulas in Lemma 8.9(a) below. We hope this will not create any confusion.

8.9. Lemma. *(a) The operators e_i, f_i on ${}^\theta\mathbf{G}_I$ are given by*

$$
e_i(M) = 1_{m-1,i}M, \quad f_i(M) = \hom_{{}^\theta\mathbf{R}_{m,1}}({}^\theta\mathbf{R}_{m+1}, M \otimes \mathbf{L}_i), \quad M \in {}^\theta\mathbf{R}_m\text{-}\mathbf{fmod}.
$$

*(b) For $M, M'' \in {}^\theta\mathbf{R}$-**mod** and $M' \in \mathbf{R}$-**mod** we have*

$$
(\psi_!(M, M') : M'') = (M \otimes M' : \psi^*(M'')).
$$

The bilinear form $(\bullet : \bullet)$ on ${}^\theta\mathbf{K}_I$ is such that

$$
(e_i(P) : P') = (1 - v^2)(P : f_i(P')), \quad P, P' \in {}^\theta\mathbf{R}\text{-}\mathbf{proj}.
$$

*(c) We have $f_i(P)^\sharp = f_i(P^\sharp)$ for each $P \in {}^\theta\mathbf{R}$-**proj**.*

*(d) We have $e_i(M)^\flat = e_i(M^\flat)$ for each $M \in {}^\theta\mathbf{R}$-**fmod**.*

(e) The operators e_i, f_i on ${}^\theta\mathbf{K}_I$, ${}^\theta\mathbf{G}_I$ satisfy the relation (7.3).

Proof : Let $M \in {}^\theta\mathbf{R}$-**fmod** and $P \in {}^\theta\mathbf{R}$-**proj**. We have $f_i(P) = \psi_!(P, \mathbf{R}_i)$. Thus

$$
\begin{aligned}
\hom_{{}^\theta\mathbf{R}}(f_i(P), M) &= \hom_{{}^\theta\mathbf{R}}(\psi_!(P, \mathbf{R}_i), M)\\
&= \hom_{{}^\theta\mathbf{R}\otimes\mathbf{R}_i}(P \otimes \mathbf{R}_i, \psi^*(M))\\
&= \hom_{{}^\theta\mathbf{R}}(P, 1_{m-1,i}M).
\end{aligned}
$$

Next, we must prove that $f_i(M) = \psi_*(M, \mathbf{L}_i)$. We have

$$\begin{aligned}
\hom_{{}^\theta\mathbf{R}}(e_i(P), M) &= \hom_{{}^\theta\mathbf{R}\otimes\mathbf{R}_i}(1_{m-1,i}P, M \otimes (\mathbf{R}_i/\varkappa_i\mathbf{R}_i)) \\
&= \hom_{{}^\theta\mathbf{R}\otimes\mathbf{R}_i}(\psi^*(P), M \otimes (\mathbf{R}_i/\varkappa_i\mathbf{R}_i)) \\
&= \hom_{{}^\theta\mathbf{R}}(P, \psi_*(M, \mathbf{L}_i)).
\end{aligned}$$

This proves part (a). The first claim of (b) follows from the following identity

$$\begin{aligned}
(\psi_!(M, M') : M'') &= \operatorname{gdim}\Big((M^\omega \otimes M'^\omega) \otimes_{{}^\theta\mathbf{R}_{m,m'}} \psi^*(M'')\Big), \\
&= (M \otimes M' : \psi^*(M'')).
\end{aligned}$$

The second one is proved as follows

$$\begin{aligned}
(1-v^2)(f_i(P) : P') &= (1-v^2)(\psi_!(P, \mathbf{R}_i) : P') \\
&= (P \otimes \mathbf{L}_i : 1_{m-1,i}P') \\
&= (P : e_i(P')).
\end{aligned}$$

Part (c) follows from the following identities

$$\begin{aligned}
f_i(P)^\sharp &= \hom_{{}^\theta\mathbf{R}_m}\big({}^\theta\mathbf{R}_m \otimes_{{}^\theta\mathbf{R}_{m-1,1}} (P \otimes \mathbf{R}_i),\, {}^\theta\mathbf{R}_m\big), \\
&= \hom_{{}^\theta\mathbf{R}_{m-1,1}}\big(P \otimes \mathbf{R}_i,\, {}^\theta\mathbf{R}_m\big), \\
&= {}^\theta\mathbf{R}_m \otimes_{{}^\theta\mathbf{R}_{m-1,1}} \hom_{{}^\theta\mathbf{R}_{m-1,1}}\big(P \otimes \mathbf{R}_i,\, {}^\theta\mathbf{R}_{m-1,1}\big), \\
&= {}^\theta\mathbf{R}_m \otimes_{{}^\theta\mathbf{R}_{m-1,1}} \big(\hom_{{}^\theta\mathbf{R}_{m-1}}(P,\, {}^\theta\mathbf{R}_{m-1}) \otimes \mathbf{R}_i\big), \\
&= f_i(P^\sharp).
\end{aligned}$$

Here the second equality is Frobenius reciprocity and the third one follows from Lemma 8.7, see e.g., [CuR, (2.29)]. Part (d) follows from (c) and Proposition 8.4(b). To prove (e) it is enough to check that the operators e_i, f_i on ${}^\theta\mathbf{K}_I$ satisfy the relation (7.3). For f_i it is enough to observe that

$$f_i(P) = \psi_!(P, \mathbf{R}_i), \quad \forall P \in {}^\theta\mathbf{R}\textbf{-proj}.$$

Then the claim follows from Theorem 7.7 and the associativity of induction. For e_i it is enough to observe that the transposed operator is given by

$$f_i(M) = \psi_*(M, \mathbf{L}_i), \quad M \in {}^\theta\mathbf{R}\textbf{-fmod}$$

and to use the associativity of coinduction.

$\square$

8.10. Shuffles, projectives, and characters. For each sequence $\mathbf{i}$ in ${}^\theta I^m$ we define a projective graded module in ${}^\theta\mathbf{R}_m$-**proj** by setting ${}^\theta\mathbf{R}_\mathbf{i} = {}^\theta\mathbf{R}_m 1_\mathbf{i}$. More generally, for $\mathbf{y} \in {}^\theta Y^m$ we define an object ${}^\theta\mathbf{R}_\mathbf{y}$ of ${}^\theta\mathbf{R}_m$-**proj** as follows. Write

$$\mathbf{y} = (\theta(\mathbf{j})\mathbf{j}, \theta(\mathbf{b})\mathbf{b}), \quad \mathbf{j} \in I^m, \quad \mathbf{b} \in \mathbb{Z}^m.$$

We may abbreviate $\mathbf{y} = \theta(\mathbf{z})\mathbf{z}$ where $\mathbf{z} = (\mathbf{j}, \mathbf{b})$. Define the idempotent $1_\mathbf{y}$ as the image of the idempotent $1_\mathbf{z}$ by the inclusion $\psi : \mathbf{R}_m \to {}^\theta\mathbf{R}_m$ given by setting m, m', m'' equal to $0, m, m$ in (8.3). Then set

$${}^\theta\mathbf{R}_\mathbf{y} = ({}^\theta\mathbf{R}_m 1_\mathbf{y})[\ell_\mathbf{b}].$$

The graded module ${}^\theta\mathbf{R}_\mathbf{y}$ satisfies the same properties as the projective graded $\mathbf{R}_m$-module $\mathbf{R}_\mathbf{z}$ introduced in Section 7.4. In particular ${}^\theta\mathbf{R}_\mathbf{y}$ is $\sharp$-selfdual, and if the sequence $\mathbf{i}$ in ${}^\theta I^m$ is the expansion of the pair $\mathbf{y}$ then we have

$$(8.5) \qquad {}^\theta\mathbf{R}_\mathbf{i} = \langle \mathbf{b} \rangle !\, {}^\theta\mathbf{R}_\mathbf{y}.$$

For $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ in ${}^\theta Y^m$ and $\mathbf{y}' = (\mathbf{i}', \mathbf{a}')$ in $Y^{m'}$ we have

$$\psi_!({}^\theta\mathbf{R}_\mathbf{y}, \mathbf{R}_{\mathbf{y}'}) = {}^\theta\mathbf{R}_{\mathbf{y}''}, \quad \mathbf{y}'' = \theta(\mathbf{y}')\mathbf{y}\mathbf{y}' = (\theta(\mathbf{i}')\mathbf{i}\mathbf{i}', \theta(\mathbf{a}')\mathbf{a}\mathbf{a}').$$

Write $\mathbf{i}' = (i'_1, \dots, i'_k)$, $\mathbf{a}' = (a'_1, \dots a'_k)$, and

$$f_{\mathbf{y}'} = f_{i'_1}^{(a'_1)} f_{i'_2}^{(a'_2)} \cdots f_{i'_k}^{(a'_k)}.$$

Lemma 8.9(a) yields

$$f_{\mathbf{y}'}(P) = \psi_!(P, \mathbf{R}_{\mathbf{y}'}), \quad P \in {}^\theta\mathbf{R}_m\text{-}\mathbf{proj}.$$

In particular, we have

$$f_{\mathbf{y}'}(\mathbf{k}) = {}^\theta\mathbf{R}_{\theta(\mathbf{y}')\mathbf{y}'}.$$

8.11. Definition. *A shuffle of a pair of sequences $(\mathbf{i}, \mathbf{i}')$ in ${}^\theta I^m \times I^{m'}$ is a sequence $\mathbf{i}''$ in ${}^\theta I^{m''}$ together with a subsequence of $\mathbf{i}''$ isomorphic to $\mathbf{i}$ and such that the complementary subsequence is equal to $\theta(\mathbf{i}')\mathbf{i}'$ modulo θ.*

Let $Sh(\mathbf{i}, \mathbf{i}')$ be the set of shuffles of $\mathbf{i}$, $\mathbf{i}'$. The assignment $w \mapsto w^{-1}(\theta(\mathbf{i}')\mathbf{i}\mathbf{i}')$ gives a bijection from $D_{m,m'}$ to $Sh(\mathbf{i}, \mathbf{i}')$. To a shuffle $\mathbf{i}''$ in $Sh(\mathbf{i}, \mathbf{i}')$ associated with an element w of $D_{m,m'}$ we assign the following degree

$$\deg(\mathbf{i}, \mathbf{i}'; \mathbf{i}'') = \deg(\sigma_{\dot{w}} 1_{\mathbf{i}''}).$$

This degree does not depend of the choice of the reduced decomposition $\dot{w}$ of w. Let ${}^\theta\mathcal{B}I^m$ be the free $\mathcal{B}$-module with basis ${}^\theta I^m$. For any f in ${}^\theta\mathcal{B}I^m$ we write

$$f = \sum_\mathbf{i} f(\mathbf{i})\, \mathbf{i}.$$

8.12. Definitions. *(a) For any finitely generated graded ${}^\theta\mathbf{R}_m$-module M we define the character of M as the element of ${}^\theta\mathcal{B}I^m$ given by*

$$\mathrm{ch}(M)=\sum_{\mathbf{i}}\mathrm{gdim}(1_{\mathbf{i}}M)\,\mathbf{i}.$$

(b) For any elements $f\in{}^\theta\mathcal{B}I^m$, $g\in\mathcal{B}I^{m'}$ we define their product $f\circledast g\in{}^\theta\mathcal{B}I^{m''}$ by

$$(f\circledast g)(\mathbf{i}'')=\sum_{\mathbf{i},\mathbf{i}'}v^{\deg(\mathbf{i},\mathbf{i}';\mathbf{i}'')}f(\mathbf{i})g(\mathbf{i}').$$

Here the sum is over all ways to represent $\mathbf{i}''$ as a shuffle of $\mathbf{i}$ and $\mathbf{i}'$.

8.13. Proposition. *For any $M\in{}^\theta\mathbf{R}_m$-**mod** and any $M'\in\mathbf{R}_{m'}$-**mod** we have*

$$\mathrm{ch}(\psi_!(M,M'))=\mathrm{ch}(M)\circledast\mathrm{ch}(M').$$

Proof : We have $\mathrm{ch}(M)=\sum_{\mathbf{i}}({}^\theta\mathbf{R}_{\mathbf{i}}:M)\,\mathbf{i}$. Thus Lemma 8.9$(b)$ yields

$$\mathrm{ch}(\psi_!(M,M'))=\sum_{\mathbf{i}''}(\psi^*({}^\theta\mathbf{R}_{\mathbf{i}''}):M\otimes M')\,\mathbf{i}''.$$

Next, we have the following formula

$$\psi^*({}^\theta\mathbf{R}_{\mathbf{i}''})=\bigoplus_{\mathbf{i},\mathbf{i}'}{}^\theta\mathbf{R}_{\mathbf{i}}\otimes\mathbf{R}_{\mathbf{i}'}[\deg(\mathbf{i},\mathbf{i}';\mathbf{i}'')],$$

where the sum runs over all sequences $\mathbf{i}$, $\mathbf{i}'$ such that $\mathbf{i}''$ lies in $Sh(\mathbf{i},\mathbf{i}')$. This formula is a consequence of the Mackey's induction-restriction theorem. The details are left to the reader. See e.g., the proof of Theorem 8.31 below. Therefore we get

$$\begin{aligned}\mathrm{ch}(\psi_!(M,M'))&=\sum_{\mathbf{i},\mathbf{i}'}\sum_{\mathbf{i}''\in Sh(\mathbf{i},\mathbf{i}')}({}^\theta\mathbf{R}_{\mathbf{i}}:M)\,(\mathbf{R}_{\mathbf{i}'}:M')\,v^{\deg(\mathbf{i},\mathbf{i}';\mathbf{i}'')}\,\mathbf{i}''\\&=\mathrm{ch}(M)\circledast\mathrm{ch}(M').\end{aligned}$$

□

8.14. Proposition. *We have*

$$f_i({}^\theta\mathbf{R}_{\mathbf{i}})={}^\theta\mathbf{R}_{\theta(i)\mathbf{i}i},\quad e_i({}^\theta\mathbf{R}_{\mathbf{i}})=\bigoplus_{\mathbf{i}'}{}^\theta\mathbf{R}_{\mathbf{i}'}[\deg(\mathbf{i}',i;\mathbf{i})].$$

Here the sum runs over all sequences in ${}^\theta I^{m-1}$ such that $\mathbf{i}$ lies in $Sh(\mathbf{i}',i)$.

Proof : Left to the reader.

□

8.15. Example. Set $\nu = i + \theta(i)$ and $\mathbf{i} = i\theta(i)$. Using the description of ${}^\theta\mathbf{R}_0$, ${}^\theta\mathbf{R}_1$ given in Example 5.9 we can compute e_j, f_j and ch. Let us regard $\mathbf{k}$ as an object of ${}^\theta\mathbf{R}_0$-**proj**. We have

$$f_i(\mathbf{k}) = {}^\theta\mathbf{R}_{\theta(\mathbf{i})}, \quad e_j({}^\theta\mathbf{R}_{\mathbf{i}}) = \begin{cases} v^{\lambda_i + \lambda_{\theta(i)}}\,\mathbf{k} & \text{if } j = i, \\ \mathbf{k} & \text{if } j = \theta(i), \\ 0 & \text{else.} \end{cases}$$

Next, observe that ${}^\theta\mathbf{R}_{0,1} = \mathbf{R}_1$ and that the inclusion in (8.3) yields following formula ${}^\theta\mathbf{R}_1 = \mathbf{R}_1 \oplus \mathbf{R}_1\pi_1$. Thus we get

$$\psi_*(\mathbf{k}, \mathbf{R}_i) = \hom_{\mathbf{R}_1}({}^\theta\mathbf{R}_1, \mathbf{R}_i) = {}^\theta\mathbf{R}_{\mathbf{i}}[\lambda_i + \lambda_{\theta(i)}],$$

$$\psi_!(\mathbf{k}, \mathbf{R}_i) = {}^\theta\mathbf{R}_1 \otimes_{\mathbf{R}_1} \mathbf{R}_i = {}^\theta\mathbf{R}_{\theta(\mathbf{i})}.$$

In particular, we have the following.

- If $\lambda_i + \lambda_{\theta(i)} \neq 0$ then $e_j({}^\theta\mathbf{L}_{\theta(\mathbf{i})}) = \mathbf{k}$ if $j = i$ and 0 else, and $f_i(\mathbf{k}) = v^{\lambda_i + \lambda_{\theta(i)}}\,{}^\theta\mathbf{L}_{\mathbf{i}} + {}^\theta\mathbf{L}_{\theta(\mathbf{i})}$. Further $\mathrm{ch}({}^\theta\mathbf{L}_{\theta(\mathbf{i})}) = \theta(\mathbf{i})$ and $\mathrm{ch}({}^\theta\mathbf{L}_{\mathbf{i}}) = \mathbf{i}$.
- If $\lambda_i + \lambda_{\theta(i)} = 0$ then $e_j({}^\theta\mathbf{L}_{\theta(\mathbf{i})}) = \mathbf{k}$ if $j = i, \theta(i)$ and 0 else, and $f_i(\mathbf{k}) = {}^\theta\mathbf{L}_{\mathbf{i}}$. Further $\mathrm{ch}({}^\theta\mathbf{L}_{\mathbf{i}}) = \mathbf{i} + \theta(\mathbf{i})$.

8.16. Induction of $\mathbf{H}_m$-modules versus induction of ${}^\theta\mathbf{R}_m$-modules. Recall the functors E_i, F_i on $\mathbf{H}$-$\mathbf{fMod}_I$ defined in (6.4). We have also the functors

$$\Psi : {}^\theta\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to \mathbf{H}_m\text{-}\mathbf{fMod}_I, \quad \mathbf{for} : {}^\theta\mathbf{R}_m\text{-}\mathbf{fmod} \to {}^\theta\mathbf{R}_m\text{-}\mathbf{fMod}_0,$$

where **for** is the forgetting of the grading. Finally we define functors

$$\begin{aligned} &E_i : {}^\theta\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to {}^\theta\mathbf{R}_{m-1}\text{-}\mathbf{fMod}_0, \quad E_iM = 1_{m-1,i}M, \\ &F_i : {}^\theta\mathbf{R}_m\text{-}\mathbf{fMod}_0 \to {}^\theta\mathbf{R}_{m+1}\text{-}\mathbf{fMod}_0, \quad F_iM = \psi_!(M, \mathbf{L}_i). \end{aligned} \tag{8.6}$$

8.17. Proposition. *There are canonical isomorphisms of functors*

$$E_i \circ \Psi = \Psi \circ E_i, \quad F_i \circ \Psi = \Psi \circ F_i, \quad E_i \circ \mathbf{for} = \mathbf{for} \circ e_i, \quad F_i \circ \mathbf{for} = \mathbf{for} \circ f_{\theta(i)}.$$

Proof: Recall that $\mathbf{k}_i$ is the 1-dimensional $\mathbf{k}[X_{m+1}^{\pm 1}]$-module such that $X_{m+1} \mapsto i^{-1}$, that $\mathbf{L}_i$ is the 1-dimensional $\mathbf{R}_1$-module such that $1_i \mapsto 1$ and $\varkappa_1 \mapsto 0$, and that Ψ identifies X_{m+1} and the element $1 \otimes i^{-1}f(\varkappa_{m+1})$ in ${}^\theta\mathbf{R}_{m,1}$, where the function f is as in Theorem 6.5. The first two isomorphisms are obvious consequences of (6.4), (8.6), because E_iM is the generalized i^{-1}-eigenspace of M with respect to the action of X_{m+1}, and F_iM is induced from the $\mathbf{H}_m \otimes \mathbf{k}[X_{m+1}^{\pm 1}]$-module $M \otimes \mathbf{k}_i$. The third isomorphism follows from (8.6) and Lemma 8.9(a). Now, we concentrate on the last isomorphism. Lemma 8.9(a) yields

$$f_{\theta(i)}(M) = \psi_*(M, \mathbf{L}_{\theta(i)}), \quad M \in {}^\theta\mathbf{R}\text{-}\mathbf{fmod}.$$

For any graded $\mathbf{R}$-module N let N^κ be equal to N, with the $\mathbf{R}$-action twisted by the involution κ in (7.1). Note that $\mathbf{L}_i^\kappa = \mathbf{L}_{\theta(i)}$. Therefore the proposition follows from the following lemma.

8.18. Lemma. *For each* $M \in {}^\theta\mathbf{R}_m$-**fmod** *and* $N \in \mathbf{R}_{m'}$-**fmod** *there is an isomorphism of (non-graded)* ${}^\theta\mathbf{R}_{m''}$*-modules* $\psi_!(M,N) = \psi_*(M,N^\kappa)$.

Proof : Recall that ${}^\theta\mathbf{R}_{m,m'} = {}^\theta\mathbf{R}_m \otimes \mathbf{R}_{m'}$. The involution $\kappa : \mathbf{R}_{m'} \to \mathbf{R}_{m'}$ in (7.1) yields an involution of ${}^\theta\mathbf{R}_{m,m'}$. Let us denote it by κ again. Let ${}^\theta\mathbf{R}^\kappa_{m,m'}$ be the $({}^\theta\mathbf{R}_{m,m'}, {}^\theta\mathbf{R}_{m,m'})$-bimodule which is equal to ${}^\theta\mathbf{R}_{m,m'}$ as a right ${}^\theta\mathbf{R}_{m,m'}$-module, and such that the left ${}^\theta\mathbf{R}_{m,m'}$-action is twisted by κ. It is enough to prove that there is an isomorphism of (non-graded) $({}^\theta\mathbf{R}_{m''}, {}^\theta\mathbf{R}_{m,m'})$-bimodules

$$ {}^\theta\mathbf{R}_{m''} \to \mathrm{hom}_{{}^\theta\mathbf{R}_{m,m'}}({}^\theta\mathbf{R}_{m''}, {}^\theta\mathbf{R}^\kappa_{m,m'}). $$

The bimodule structure on the right hand side is given by

$$ (xfy)(z) = f(zx)y, \quad x, z \in {}^\theta\mathbf{R}_{m''},\ y \in {}^\theta\mathbf{R}_{m,m'}. $$

Lemma 8.7 yields an isomorphism

$$ {}^\theta\mathbf{R}_{m''} = \bigoplus_{w \in D_{m,m'}} {}^\theta\mathbf{R}_{m,m'}\sigma_{\dot w} $$

of graded ${}^\theta\mathbf{R}_{m,m'}$-modules. The longest double coset representative in $D_{m,m';m,m'}$ is the coset of the involution $u \in W_{m''}$ given by

$$ u = w_{m'}\,\varepsilon_{m+1}\ldots\varepsilon_{m''}, $$

with $w_{m'}$ the longest element of $\mathfrak{S}_{m'}$. There is an unique morphism of $({}^\theta\mathbf{R}_{m,m'}, {}^\theta\mathbf{R}_{m,m'})$-bimodules

$$ h : {}^\theta\mathbf{R}_{m,m'} \to \mathrm{hom}_{{}^\theta\mathbf{R}_{m,m'}}({}^\theta\mathbf{R}_{m''}, {}^\theta\mathbf{R}^\kappa_{m,m'}), $$

taking 1 to the map

$$ y\,\sigma_{\dot w} \mapsto \kappa(y)\,\delta_{w,u}, \quad y \in {}^\theta\mathbf{R}_{m,m'}, \quad w \in D_{m,m';m,m'}. $$

Since the right hand side is a left ${}^\theta\mathbf{R}_{m''}$-module, by Frobenius reciprocity h yields a morphism of $({}^\theta\mathbf{R}_{m''}, {}^\theta\mathbf{R}_{m,m'})$-bimodules

$$ {}^\theta\mathbf{R}_{m''} \to \mathrm{hom}_{{}^\theta\mathbf{R}_{m,m'}}({}^\theta\mathbf{R}_{m''}, {}^\theta\mathbf{R}^\kappa_{m,m'}). $$

This map is invertible. The proof is the same as in [M, sec. 3], [LV, thm. 2.2].

□

8.19. Proposition. *(a) The functor* Ψ *yields an isomorphism of Abelian groups*

$$ \bigoplus_{m \geqslant 0} [{}^\theta\mathbf{R}_m\text{-}\mathbf{fMod}_0] = \bigoplus_{m \geqslant 0} [\mathbf{H}_m\text{-}\mathbf{fMod}_I]. $$

The functors E_i*,* F_i *yield endomorphisms of both sides which are intertwined by* Ψ*.*

(b) The forgetful functor **for** *factors to a group isomorphism*

$$ {}^\theta\mathbf{G}_I/(v-1) = \bigoplus_{m \geqslant 0} [{}^\theta\mathbf{R}_m\text{-}\mathbf{fMod}_0]. $$

Proof : Claim (a) follows from Corollary 6.7 and Proposition 8.17. Claim (b) follows from Proposition 8.2.

□

8.20. The crystal operators on ${}^\theta\mathbf{G}_I$ and ${}^\theta B(\lambda)$. Fix a vertex i in I. For each irreducible graded module $M \in {}^\theta\mathbf{R}\text{-}\mathbf{fmod}$ we define

$$\tilde{e}_i(M) = \mathrm{soc}\,(e_i(M)), \quad \tilde{f}_i(M) = \mathrm{top}\,\psi_!(M, \mathbf{L}_i),$$
$$\varepsilon_i(M) = \max\{n \geqslant 0; e_i^n(M) \neq 0\}, \quad \varepsilon_i(M) \in \mathbb{N} \cup \{\infty\}.$$

For each positive integers $m \geqslant n$ we consider the functor

$$\Delta_{ni} : {}^\theta\mathbf{R}_m\text{-}\mathbf{fmod} \to {}^\theta\mathbf{R}_{m-n}\text{-}\mathbf{fmod} \times \mathbf{R}_{ni}\text{-}\mathbf{fmod}, \quad M \mapsto 1_{m-n,ni}M.$$

Given an irreducible graded module $M \in {}^\theta\mathbf{R}_m\text{-}\mathbf{fmod}$ we have, see Lemma 8.9(a),

$$\varepsilon_i(M) = \max\{n \geqslant 0; \Delta_{ni}(M) \neq 0\}, \quad e_i(M) = \Delta_i(M).$$

8.21. Proposition. *Let M be an irreducible graded ${}^\theta\mathbf{R}_m$-module and n be an integer $\geqslant 0$. Set $\varepsilon = \varepsilon_i(M)$, $M^+ = \psi_!(M, \mathbf{L}_{ni})$ and $M^- = \Delta_{ni}(M)$.*

(a) If $\varepsilon = 0$ then $\Delta_{ni}(M^+) = M \otimes \mathbf{L}_{ni}$, $\mathrm{top}(M^+)$ is irreducible, $\varepsilon_i(\mathrm{top}(M^+)) = n$, all other composition factors L of M^+ have $\varepsilon_i(L) < n$.

(b) If $\varepsilon \geqslant n$ then any irreducible submodule of M^- is of the form $N \otimes \mathbf{L}_{ni}$ with $\varepsilon_i(N) = \varepsilon - n$. If $\varepsilon = n$ then M^- is irreducible. If $\varepsilon \geqslant n$ then $\mathrm{soc}(M^-)$ is irreducible. In particular $\tilde{e}_i(M)$ is irreducible if $\varepsilon \neq 0$ and 0 else. Finally we have $\mathrm{soc}(M^-) = \tilde{e}_i^n(M) \otimes \mathbf{L}_{ni}$.

(c) $\mathrm{top}(M^+)$ is irreducible, $\varepsilon_i(\mathrm{top}(M^+)) = \varepsilon + n$, and all other composition factors L of M^+ have $\varepsilon_i(L) < \varepsilon + n$. In particular $\tilde{f}_i(M)$ is irreducible. Finally we have $\mathrm{top}(M^+) = \tilde{f}_i^n(M)$.

Proof : Part (a) is the analogue of [K, lem. 5.1.3], [KL, lem. 3.7]. More precisely, note first that we have

$$\mathrm{ch}(\Delta_{ni}(M^+)) = \sum_{\mathbf{i}} \mathrm{gdim}(1_{\theta(i^n)\mathbf{i}i^n} M^+)\, \theta(i^n)\mathbf{i}i^n. \tag{8.7}$$

Hence, since $\varepsilon = 0$ Proposition 8.13 implies that

$$\dim(\Delta_{ni}(M^+)) = \dim(M \otimes \mathbf{L}_{ni}).$$

Since $\Delta_{ni}(M^+)$ contains a copy of $M \otimes \mathbf{L}_{ni}$, we get the first claim of (a). By Frobenius reciprocity, a copy of $M \otimes \mathbf{L}_{ni}$, possibly with a grading shift, appears as a submodule of $\Delta_{ni}(M')$ for any nonzero quotient $M^+ \to M'$. Since

$$\Delta_{ni}(M^+) = M \otimes \mathbf{L}_{ni},$$

this implies that $\mathrm{top}(M^+)$ is irreducible with $\varepsilon_i(\mathrm{top}(M^+)) \geqslant n$, that

$$\Delta_{ni}(M^+) = \Delta_{ni}(\mathrm{top}(M^+)),$$

and that $\Delta_{ni}(L) = 0$ for all other composition factors L of M^+. Finally we have $\varepsilon_i(\mathrm{top}(M^+)) = n$, because $\varepsilon = 0$.

Now we prove (b). The first claim is the analogue of [K, lem. 5.1.2]. Indeed, any irreducible submodule of M^- is of the form $N \otimes \mathbf{L}_{ni}$ with N irreducible. We have

$\varepsilon_i(N) \leqslant \varepsilon - n$ by definition of ε_i. For the reverse inequality, Frobenius reciprocity and the irreducibility of M imply that M is a quotient of $\psi_!(N, \mathbf{L}_{ni})$. So applying the exact functor $\Delta_{\varepsilon i}$ we see that $\Delta_{\varepsilon i}(M)$ is a quotient of $\Delta_{\varepsilon i}\psi_!(N, \mathbf{L}_{ni})$. In particular

$$\Delta_{\varepsilon i}\psi_!(N, \mathbf{L}_{ni}) \neq 0.$$

By Proposition 8.13 and (8.7) we have also $\Delta_{(\varepsilon-n)i}(N) \neq 0$. Thus $\varepsilon_i(N) = \varepsilon - n$. The second claim of ($b$) is the analogue of [K, lem. 5.1.4]. Indeed, if $\varepsilon = n$ then any irreducible submodule of M^- is of the form $N \otimes \mathbf{L}_{ni}$ with $\varepsilon_i(N) = 0$. Once again Frobenius reciprocity and the irreducibility of M imply that M is a quotient of $\psi_!(N, \mathbf{L}_{ni})$. Hence M^- is a quotient of $\Delta_{ni}\psi_!(N, \mathbf{L}_{ni})$. But the later is isomorphic to $N \otimes \mathbf{L}_{ni}$ by (a). Next, the third claim of (b) is the analogue of [K, lem. 5.1.6], [KL, prop. 3.10]. Indeed, suppose that $N \otimes \mathbf{L}_{ni} \subset \mathrm{soc}(M^-)$. Then $\varepsilon_i(N) = \varepsilon - n$ by the first part of (b). Thus N contributes a non-trivial submodule to $\Delta_{\varepsilon i}(M)$. But $\Delta_{\varepsilon i}(M)$ is an irreducible graded ${}^\theta\mathbf{R}_{m-\varepsilon,\varepsilon}$-module by the second part of (b). Thus the socle of $\Delta_{\varepsilon i}(M)$ as a graded ${}^\theta\mathbf{R}_{m-\varepsilon,\varepsilon-n,n}$-module is $N \otimes \mathbf{L}_{(\varepsilon-n)i} \otimes \mathbf{L}_{ni}$ by Example 7.5. Hence $\mathrm{soc}(M^-)$ must equal $N \otimes \mathbf{L}_{ni}$. Finally, the last claim of (b) is the analogue of [K, lem. 5.2.1(i)], [KL, lem. 3.13]. Indeed, note first that if $n > \varepsilon$ then

$$\mathrm{soc}(M^-) = \tilde{e}_i^n(M) = 0.$$

Assume now that $\varepsilon \geqslant n$. Observe that

$$\tilde{e}_i(M) = \mathrm{soc}(\Delta_i(M))$$

is irreducible or zero by the third part of (b). Hence $\tilde{e}_i(M) \otimes \mathbf{L}_i$ is a submodule of $\Delta_i(M)$. Applying this n times we deduce that $\tilde{e}_i^n(M) \otimes (\mathbf{L}_i)^{\otimes n}$ is a submodule of $\Delta_{ni}(M)$ as a graded ${}^\theta\mathbf{R}_{m-n,1^n}$-module. Hence $\tilde{e}_i^n(M) \otimes \mathbf{L}_{ni}$ is a submodule of $\Delta_{ni}(M)$ by Frobenius reciprocity.

Finally, we prove (c). It is the analogue of [K, lem. 5.2.1(ii)], [KL, lem. 3.13]. Indeed, by (b) the graded module $\Delta_{\varepsilon i}(M)$ is of the form $N \otimes \mathbf{L}_{ni}$, with N irreducible such that $\varepsilon_i(N) = 0$. Thus, by Frobenius reciprocity M is a quotient of $\psi_!(N, \mathbf{L}_{\varepsilon i})$. So the transitivity of induction implies that M^+ is a quotient of $\psi_!(N, \mathbf{L}_{(\varepsilon+n)i})$. Hence all claims except the last one follow from (a). Finally, by exactness of the induction $\tilde{f}_i^n(M)$ is a quotient of M^+, hence they are equal by simplicity of the top.

□

For each irreducible module b in ${}^\theta B(\lambda)$ we define

$$\tilde{E}_i(b) = \mathrm{soc}(E_i b), \quad \tilde{F}_i(b) = \mathrm{top}(F_i b), \quad \varepsilon_i(b) = \max\{n \geqslant 0; E_i^n b \neq 0\}. \tag{8.8}$$

Hence, we have

$$\mathbf{for} \circ \tilde{e}_i = \tilde{E}_i \circ \mathbf{for}, \quad \mathbf{for} \circ \tilde{f}_i = \tilde{F}_i \circ \mathbf{for}, \quad \varepsilon_i = \varepsilon_i \circ \mathbf{for}.$$

8.22. Proposition. *For each b, b' in ${}^\theta B(\lambda)$ we have*

(a) $\tilde{F}_i(b) \in {}^\theta B(\lambda)$,

(b) $\tilde{E}_i(b) \in {}^\theta B(\lambda) \cup \{0\}$,

(c) $\tilde{F}_i(b) = b' \iff \tilde{E}_i(b') = b$,

(d) $\varepsilon_i(b) = \max\{n \geqslant 0; \tilde{E}_i^n(b) \neq 0\}$,

(e) $\varepsilon_i(\tilde{F}_i(b)) = \varepsilon_i(b) + 1$,

(f) if $\tilde{E}_i(b) = 0$ *for all* i *then* $b = \mathbf{k}$.

Proof : Parts (a), (b), (d), (e) and (f) are immediate consequences of Proposition 8.21. Part (c) is proved as in [K, lem. 5.2.3]. More precisely, let M, N be irreducible graded modules. By Proposition 8.21(c) we have

$$\begin{aligned}\tilde{f}_i(M) = N &\iff \mathrm{Hom}_{{}^\theta\mathbf{R}}(\psi_!(M, \mathbf{L}_i), N) \neq 0 \\ &\iff \mathrm{Hom}_{{}^\theta\mathbf{R}}(M \otimes \mathbf{L}_i, e_i(N)) \neq 0 \\ &\iff \mathrm{Hom}_{{}^\theta\mathbf{R}}(M \otimes \mathbf{L}_i, \mathrm{soc}(e_i N)) \neq 0 \\ &\iff M = \tilde{e}_i(N).\end{aligned}$$

Note that the proposition can also be deduced from [M, Section 4] and Proposition 8.17. ◻

8.23. Proposition. *The following identity holds in* ${}^\theta\mathbf{K}_I$

$$f_i {}^\theta G^{\mathrm{low}}(b) = \langle \varepsilon_i(b) + 1 \rangle {}^\theta G^{\mathrm{low}}(\tilde{F}_i b) + \sum_{b'} f_{b,b'} {}^\theta G^{\mathrm{low}}(b'), \quad \forall b \in {}^\theta B(\lambda),$$

where b' *runs over the elements of* ${}^\theta B(\lambda)$ *such that* $\varepsilon_i(b') > \varepsilon_i(b) + 1$*, and* $f_{b,b'} \in \mathcal{A}$*.*

Proof : We claim that there are elements $f_{b',b}$ in $\mathcal{A}$ such that

$$e_i {}^\theta G^{\mathrm{up}}(b) = \langle \varepsilon_i(b) \rangle {}^\theta G^{\mathrm{up}}(\tilde{E}_i b) + \sum_{b'} f_{b',b} {}^\theta G^{\mathrm{up}}(b'), \tag{8.9}$$

where b' runs over the elements of ${}^\theta B(\lambda)$ with $\varepsilon_i(b') < \varepsilon_i(b) - 1$. Taking the transpose with respect to the Cartan pairing, Proposition 8.4(a), Definition 8.8, and Proposition 8.22 yield

$$\begin{aligned}f_i {}^\theta G^{\mathrm{low}}(b) &= \langle \varepsilon_i(\tilde{F}_i b) \rangle {}^\theta G^{\mathrm{low}}(\tilde{F}_i b) + \sum_{b'} f_{b,b'} {}^\theta G^{\mathrm{low}}(b'), \\ &= \langle \varepsilon_i(b) + 1 \rangle {}^\theta G^{\mathrm{low}}(\tilde{F}_i b) + \sum_{b'} f_{b,b'} {}^\theta G^{\mathrm{low}}(b'),\end{aligned}$$

where $b' \in {}^\theta B(\lambda)$ with $\varepsilon_i(b) + 1 < \varepsilon_i(b')$. Now, let us prove (8.9). This is the analogue of [K, lem. 5.5.1(i)]. Fix an irreducible ${}^\theta\mathbf{R}_m$-module b. Set

$$\varepsilon = \varepsilon_i(b), \quad M = {}^\theta G^{\mathrm{up}}(b), \quad N = {}^\theta G^{\mathrm{up}}(\tilde{E}_i^\varepsilon b).$$

We can assume that $\varepsilon > 0$, because else (8.9) is obvious. Note that $\varepsilon_i(N) = 0$ by Proposition 8.22. By Frobenius reciprocity and Proposition 8.21(b) there is a short exact sequence of graded modules

$$0 \to R \to \psi_!(N, \mathbf{L}_{\varepsilon i}) \to M \to 0.$$

Applying the functor e_i we obtain the following exact sequence of graded modules

$$0 \to e_i R \to e_i \psi_!(N, \mathbf{L}_{\varepsilon i}) \to e_i M \to 0.$$

Note that

$$e_i \psi_!(N, \mathbf{L}_{\varepsilon i}) = 1_{m-1,i}{}^\theta \mathbf{R}_m 1_{m-\varepsilon,\varepsilon i} \otimes_{{}^\theta \mathbf{R}_{m-\varepsilon,\varepsilon}} (N \otimes \mathbf{L}_{\varepsilon i}).$$

Note also that

$$D_{m-1,1;\, m-\varepsilon,\varepsilon} = \{e, x, y\},$$

$$x = s_{m-1} \dots s_{m-\varepsilon+1} s_{m-\varepsilon}, \quad y = s_{m-1} \dots s_{m-\varepsilon+1} \varepsilon_{m-\varepsilon+1},$$

$$W(e) = W_{m-\varepsilon,\varepsilon-1,1}, \quad W(x) = W_{m-\varepsilon-1,1,\varepsilon}, \quad W(y) = W_{m-\varepsilon,\varepsilon-1,1}.$$

By Proposition 5.5 we can filter the graded $({}^\theta \mathbf{R}_{m-1,1}, {}^\theta \mathbf{R}_{m-\varepsilon,\varepsilon})$-bimodule

$$1_{m-1,i}{}^\theta \mathbf{R}_m 1_{m-\varepsilon,\varepsilon i}.$$

This filtration is the same as in the Mackey induction-restriction theorem. Compare Lemma 8.32 below and the references there. The associated graded is a direct sum of graded $({}^\theta \mathbf{R}_{m-1,1}, {}^\theta \mathbf{R}_{m-\varepsilon,\varepsilon})$-bimodules labelled by elements of $\{e, x, y\}$

$$\mathrm{gr}(1_{m-1,i}{}^\theta \mathbf{R}_m 1_{m-\varepsilon,\varepsilon i}) = P_e \oplus P_x \oplus P_y.$$

We have

$$P_x 1_{m-\varepsilon-1,\varepsilon i+i} = P_x.$$

Thus, since $e_i(N) = 1_{m-\varepsilon-1,i} N = 0$, we have also

$$P_x \otimes_{{}^\theta \mathbf{R}_{m-\varepsilon,\varepsilon}} (N \otimes \mathbf{L}_{\varepsilon i}) = 0.$$

Next, we have

$$P_y 1_{m-\varepsilon,\theta(i),\varepsilon i-i} = P_y.$$

Since $1_\nu \mathbf{L}_{\varepsilon i} = 0$ if $\nu \neq \varepsilon i$ we have also

$$P_y \otimes_{{}^\theta \mathbf{R}_{m-\varepsilon,\varepsilon}} (N \otimes \mathbf{L}_{\varepsilon i}) = 0.$$

Finally, we have

$$P_e = 1_{m-1,i}{}^\theta \mathbf{R}_{m-1,1} \otimes_{\mathbf{R}'} {}^\theta \mathbf{R}_{m-\varepsilon,\varepsilon} 1_{m-\varepsilon,\varepsilon i}, \quad \mathbf{R}' = {}^\theta \mathbf{R}_{m-\varepsilon,\varepsilon-1,1}.$$

Therefore, we obtain

$$e_i \psi_!(N, \mathbf{L}_{\varepsilon i}) = {}^\theta \mathbf{R}_{m-1,1} \otimes_{\mathbf{R}'} (N \otimes \mathbf{L}_{\varepsilon i}).$$

By Example 7.5 the graded $\mathbf{R}_{\varepsilon-1,1}$-module $\mathbf{L}_{\varepsilon i}$ has a filtration by graded submodules whose associated graded is isomorphic to

$$\langle \varepsilon \rangle \mathbf{L}_{(\varepsilon-1)i} \otimes \mathbf{L}_i.$$

Therefore, up to some filtration, we have

$$e_i\psi_!(N,\mathbf{L}_{\varepsilon i}) = \langle\varepsilon\rangle\psi_!(N,\mathbf{L}_{(\varepsilon-1)i})\otimes\mathbf{L}_i.$$

Now, by Proposition 8.21$(a),(c)$ we have

$$\mathrm{top}\,\psi_!(N,\mathbf{L}_{(\varepsilon-1)i}) = \tilde f_i^{\varepsilon-1}(N) = \tilde e_i(M)$$

and all other composition factors L of $\psi_!(N,\mathbf{L}_{(\varepsilon-1)i})$ have $\varepsilon_i(L)<\varepsilon-1$. Moreover, by Proposition 8.21(a) all composition factors L of R have $\varepsilon_i(L)<\varepsilon$. Thus, by Proposition 8.21(b) all composition factors of $e_i(R)$ are of the form $L\otimes\mathbf{L}_i$ with $\varepsilon_i(L)<\varepsilon-1$. Therefore, we obtain

$$e_i(M) = \langle\varepsilon\rangle\,\tilde e_i(M) + \sum_r f_r N_r,\quad f_r\in\mathcal{A},$$

where N_r is an irreducible graded module with $\varepsilon_i(N_r)<\varepsilon-1$.

□

8.24. Example. Set $\nu = i+\theta(i)$ and $\mathbf{i} = i\theta(i)$. Let us compute $\tilde e_j$ and ε_j.

- If $\lambda_i+\lambda_{\theta(i)}\neq 0$ then $\tilde e_j({}^\theta\mathbf{L}_\mathbf{i}) = \mathbf{k}$ if $j=\theta(i)$ and 0 else. We have $\varepsilon_j({}^\theta\mathbf{L}_\mathbf{i})=1$ if $j=\theta(i)$ and 0 else.
- If $\lambda_i+\lambda_{\theta(i)}=0$ then $\tilde e_j({}^\theta\mathbf{L}_\mathbf{i}) = \mathbf{k}$ if $j=i,\theta(i)$ and 0 else. We have $\varepsilon_j({}^\theta\mathbf{L}_\mathbf{i})=1$ if $j=i,\theta(i)$ and 0 else.

8.25. Remark. If M is an irreducible graded ${}^\theta\mathbf{R}_m$-module such that $\varepsilon_i(M)=m$ then Proposition 8.21(b) implies that $\tilde e_i^m(M)=\mathbf{k}$ and $\Delta_{mi}(M)=\mathbf{k}\otimes\mathbf{L}_{mi}$. Further, by Proposition 8.22(c) there is a unique M as above, up to isomorphism, such that $\tilde e_i^m(M)=\mathbf{k}$. We claim that $M\simeq{}^\theta\mathbf{L}_{mi}$, the top of ${}^\theta\mathbf{R}_{mi}$. By Proposition 8.21(a) we have $M=\mathrm{top}(\psi_!(\mathbf{k},\mathbf{L}_{mi}))$. First, recall that ${}^\theta\mathbf{R}_{mi}=\psi_!(\mathbf{k},\mathbf{R}_{mi})$. Thus, since $\psi_!$ is exact, there is a surjective map ${}^\theta\mathbf{L}_{mi}\to M$. So it is enough to check that ${}^\theta\mathbf{L}_{mi}$ is irreducible (left to the reader). We'll not need this.

8.26. Proof of Proposition 8.2. First, we prove the following.

8.27. Proposition. *The character map* $\mathrm{ch}:{}^\theta\mathbf{G}_{I,m}\to{}^\theta\mathcal{B}I^m$ *is injective.*

Proof : The proof is similar to that of [K, thm. 5.3.1]. We must prove that the characters of the irreducible graded modules in ${}^\theta\mathbf{R}_m$**-fmod** are linearly independent. We proceed by induction on m, the case $m=0$ being trivial. Suppose $m>0$ and there is a non-trivial $\mathcal{A}$-linear dependence

$$\sum_M c_M\,\mathrm{ch}(M)=0. \tag{8.10}$$

We'll show by downward induction on $\varepsilon_i(M)$ that $c_M=0$ for each graded ${}^\theta\mathbf{R}_m$-module M which enter in (8.10). Fix M as above. We have $\varepsilon_i(M)\leqslant m$. First, assume that $\varepsilon_i(M)=m$. Note that M is the unique irreducible graded ${}^\theta\mathbf{R}_m$-module such that $\Delta_{mi}(M)=0$. Indeed we have $M={}^\theta\mathbf{L}_{mi}$, see Remark 8.25. Applying Δ_{mi} to the ${}^\theta\mathbf{R}_m$-modules which enter in (8.10) and using the formula

$$\mathrm{ch}(\Delta_{mi}(M)) = \sum_\mathbf{i}\mathrm{gdim}(1_{\theta(i^m)\mathbf{i}i^m}M)\,\theta(i^m)\mathbf{i}i^m,$$

we deduce that the coefficient c_M is zero. Now, assume that $\varepsilon_i(M) = k < m$ and that we have shown that $c_N = 0$ for all N with $\varepsilon_i(N) > k$. Applying Δ_{ki} to the ${}^\theta\mathbf{R}_m$-modules which enter in (8.10) we get

$$\sum_N c_N \operatorname{ch}(\Delta_{ki} N) = 0,$$

where N runs over all irreducible graded ${}^\theta\mathbf{R}_m$-modules with $\varepsilon_i(N) = k$. For such a graded module N we have $\Delta_{ki}(N) = \tilde{e}_i^k(N) \otimes \mathbf{L}_{ki}$ by Proposition 8.21(b). Further $\tilde{e}_i^k(N) \neq \tilde{e}_i^k(N')$ if $N \neq N'$ by Proposition 8.22(c). So we conclude by the induction hypothesis.

□

Now, we can prove Proposition 8.2. Forgetting the grading takes irreducible graded ${}^\theta\mathbf{R}$-modules to irreducible modules, and any irreducible module in ${}^\theta\mathbf{R}$-**fMod**$_0$ comes from an irreducible graded module in ${}^\theta\mathbf{R}$-**fmod** which is unique up to isomorphism and up to grading shift, see e.g., [NV, thm. 4.4.4(v), thm. 9.6.8]. Thus it is enough to prove that for any irreducible graded module M there is an integer d such that $M[d]$ is $\flat$-selfdual. This is proved as in [KL, p. 342]. More precisely, by definition of the duality functor $\flat$, for any graded module M in ${}^\theta\mathbf{R}$-**fmod** we have

$$\operatorname{ch}(M^\flat) = \operatorname{ch}(M) \bmod (v-1), \quad \operatorname{gdim}(1_\mathbf{i} M^\flat) = \overline{\operatorname{gdim}(1_\mathbf{i} M)}.$$

Thus if M is irreducible then we have $M^\flat = M[d]$ for some integer d. We must prove that d is even. It is enough to prove the following.

8.28. Lemma. *If $M \in {}^\theta\mathbf{R}$-**fmod** is irreducible then for each $\mathbf{i}$ we have*

$$\operatorname{gdim}(1_\mathbf{i} M) \in v\mathbb{Z}[v^2, v^{-2}] \cup \mathbb{Z}[v^2, v^{-2}].$$

Proof : Indeed, we'll prove that this identity holds for the projective module $M = {}^\theta\mathbf{R}_\mathbf{j}$ where $\mathbf{j}$ is any sequence in ${}^\theta I^m$. This implies our claim. Set $\mathbf{j} = (j_{1-m}, \dots, j_{m-1}, j_m)$. Proposition 8.13 yields

$$\operatorname{ch}({}^\theta\mathbf{R}_\mathbf{j}) = \operatorname{ch}({}^\theta\mathbf{R}_{j_0 j_1}) \circledast \operatorname{ch}(\mathbf{R}_{j_2}) \circledast \cdots \circledast \operatorname{ch}(\mathbf{R}_{j_m}).$$

Examples 5.9(b), 7.5(a) yield

$$\operatorname{ch}({}^\theta\mathbf{R}_{j_0 j_1}) = (1-v^2)^{-1}\big(j_0 j_1 + v^{\lambda_{j_0}+\lambda_{j_1}} j_1 j_0\big), \quad \operatorname{ch}(\mathbf{R}_{j_k}) = (1-v^2)^{-1}\, j_k.$$

So, by Definition 8.12(b), it is enough to check that for each reflection w of W_m which fixes the sequence $\mathbf{j}$ the degree of $\sigma_{\dot w} 1_\mathbf{j}$ is even. This reduces to the following computation (left to the reader). Fix $k \neq l$ such that $j_k = j_l$. If one of the following holds

$$1 \leqslant k < l, \quad \dot w = s_k \dots s_{l-2} s_{l-1} s_{l-2} \dots s_k,$$

$$1 \leqslant 1-k < l, \quad \dot w = s_{-k} \cdots s_1 \varepsilon_1 s_1 \dots s_{l-1} \dots s_1 \varepsilon_1 s_1 \dots s_{-k},$$

then $\deg(\sigma_{\dot w} 1_\mathbf{j})$ is even.

□

8.29. The algebra ${}^\theta\mathbf{B}$ and its representation in ${}^\theta\mathbf{V}(\lambda)$. Following [EK1,2,3] we define a $\mathcal{K}$-algebra ${}^\theta\mathbf{B}$ as follows.

8.30. Definition. *Let ${}^\theta\mathbf{B}$ be the $\mathcal{K}$-algebra generated by e_i, f_i and invertible elements t_i, $i \in I$, satisfying the following defining relations*

(*a*) $t_i t_j = t_j t_i$ *and* $t_{\theta(i)} = t_i$ *for all* i, j,

(*b*) $t_i e_j t_i^{-1} = v^{i\cdot j + \theta(i)\cdot j} e_j$ *and* $t_i f_j t_i^{-1} = v^{-i\cdot j - \theta(i)\cdot j} f_j$ *for all* i, j,

(*c*) $e_i f_j = v^{-i\cdot j} f_j e_i + \delta_{i,j} + \delta_{\theta(i),j} t_i$ *for all* i, j,

(*d*) *formula* (7.3) *holds with* $\theta_i = e_i$, *or with* $\theta_i = f_i$.

We define a representation of ${}^\theta\mathbf{B}$ as follows. The $\mathcal{K}$-vector space

$$ {}^\theta\mathbf{V}(\lambda) = \mathcal{K} \otimes_{\mathcal{A}} {}^\theta\mathbf{K}_I $$

is equipped with $\mathcal{K}$-linear operators e_i, e'_i, f_i and with a $\mathcal{K}$-bilinear form in the obvious way. Let ϕ_λ be the class of $\mathbf{k}$ in ${}^\theta\mathbf{K}_I$, where $\mathbf{k}$ is regarded as the trivial module over the $\mathbf{k}$-algebra ${}^\theta\mathbf{R}_0$. Let λ be as in (6.2). We can now prove the following theorem, which is the main result of Section 8.

8.31. Theorem. *(a) The operators e_i, f_i define a representation of ${}^\theta\mathbf{B}$ on ${}^\theta\mathbf{V}(\lambda)$. The ${}^\theta\mathbf{B}$-module ${}^\theta\mathbf{V}(\lambda)$ is irreducible and for $i \in I$ we have*

$$ e_i\phi_\lambda = 0, \quad t_i\phi_\lambda = v^{\lambda_i + \lambda_{\theta(i)}}\phi_\lambda, \quad \{x \in {}^\theta\mathbf{V}(\lambda);\, e_j x = 0, \forall j\} = \mathcal{K}\,\phi_\lambda. $$

(b) There is a unique symmetric $\mathcal{K}$-bilinear form $(\bullet : \bullet)$ on ${}^\theta\mathbf{V}(\lambda)$ such that $(\phi_\lambda : \phi_\lambda) = 1$ and $(e'_i x : y) = (x : f_i y)$ for all $i \in I$, $x, y \in {}^\theta\mathbf{V}(\lambda)$, and it is non-degenerate.

(c) There is a unique $\mathcal{K}$-antilinear map ${}^\theta\mathbf{V}(\lambda) \to {}^\theta\mathbf{V}(\lambda)$ such that $P \mapsto P^\sharp$ for all graded projective module P. It is the unique $\mathcal{K}$-antilinear map such that $\phi_\lambda^\sharp = \phi_\lambda$ and $(f_i x)^\sharp = f_i(x^\sharp)$ for all $x \in {}^\theta\mathbf{V}(\lambda)$.

Proof : For each i in I we define the $\mathcal{A}$-linear operator t_i on ${}^\theta\mathbf{K}_I$ by setting

$$ t_i P = v^{\lambda_i + \lambda_{\theta(i)} - \nu\cdot(i+\theta(i))} P, \quad \forall P \in {}^\theta\mathbf{R}_\nu\text{-}\mathbf{proj}. $$

We must prove that the operators e_i, f_i, and t_i satisfy the relations in Definition 8.30. Relation (c) is the only non trivial one, see Lemma 8.9(e). To check it we need a version of the Mackey's induction-restriction theorem. Note that we have

$$ D_{m,1;m,1} = \{e, s_m, \varepsilon_{m+1}\}, $$
$$ W(e) = W_{m,1}, \quad W(s_m) = W_{m-1,1,1}, \quad W(\varepsilon_{m+1}) = W_{m,1}. $$

8.32. Lemma. *Fix i, j in I. Let μ, ν in ${}^\theta\mathbb{N}I$ be such that $\nu + i + \theta(i) = \mu + j + \theta(j)$. Put $m = |\nu|/2 = |\mu|/2$. The graded $({}^\theta\mathbf{R}_{m,1}, {}^\theta\mathbf{R}_{m,1})$-bimodule $1_{\nu,i}{}^\theta\mathbf{R}_{m+1}1_{\mu,j}$ has a filtration by graded bimodules whose associated graded is isomorphic to :*

(a) $\big({}^\theta\mathbf{R}_\nu \otimes \mathbf{R}_i\big) \oplus \Big(({}^\theta\mathbf{R}_m 1_{\nu',i} \otimes \mathbf{R}_i) \otimes_{\mathbf{R}'} (1_{\nu',i}{}^\theta\mathbf{R}_m \otimes \mathbf{R}_i)\Big)[d]$ *if* $j = i$,

(b) $\big({}^\theta\mathbf{R}_\nu \otimes \mathbf{R}_{\theta(i)}\big)[d'] \oplus \Big(({}^\theta\mathbf{R}_m 1_{\nu',\theta(i)} \otimes \mathbf{R}_i) \otimes_{\mathbf{R}'} (1_{\nu',i}{}^\theta\mathbf{R}_m \otimes \mathbf{R}_{\theta(i)})\Big)[d]$ *if* $j = \theta(i)$,

(c) $\Big(({}^\theta\mathbf{R}_m 1_{\nu',j}\otimes\mathbf{R}_i)\otimes_{\mathbf{R}'}(1_{\nu',i}{}^\theta\mathbf{R}_m\otimes\mathbf{R}_j)\Big)[d]$ *if* $j\neq i,\theta(i)$.

Here we have set $\nu'=\nu-j-\theta(j)$, $\mathbf{R}'={}^\theta\mathbf{R}_{m-1,1,1}$, $d=\deg(\sigma_m 1_{\nu',i,j})$, *and* $d'=\deg(\pi_{m+1}1_{\nu,\theta(i)})$

The proof is similar to the proof of [M, thm. 1], [KL, prop. 2.18]. It is left to the reader. Note that we have the following formulas, see Remark 5.2,

$$\deg(\pi_{m+1}1_{\nu,\theta(i)})=\lambda_i+\lambda_{\theta(i)}-\nu\cdot(i+\theta(i))/2,\quad \deg(\sigma_m 1_{\nu',i,j})=-i\cdot j.$$

Now, recall that P lie in ${}^\theta\mathbf{R}_\nu$**-proj** and that

$$f_j(P)={}^\theta\mathbf{R}_{m+1}1_{m,j}\otimes_{{}^\theta\mathbf{R}_{m,1}}(P\otimes\mathbf{R}_1),\quad e_i'(P)=1_{m-1,i}P,$$

where $1_{m-1,i}P$ is regarded as a ${}^\theta\mathbf{R}_{m-1}$-module. Therefore we have

$$e_i'f_j(P)=1_{m,i}{}^\theta\mathbf{R}_{m+1}1_{m,j}\otimes_{{}^\theta\mathbf{R}_{m,1}}(P\otimes\mathbf{R}_1),$$
$$f_je_i'(P)={}^\theta\mathbf{R}_m 1_{m-1,j}\otimes_{{}^\theta\mathbf{R}_{m-1,1}}(1_{m-1,i}P\otimes\mathbf{R}_1).$$

Therefore we have the following identities

- $e_i'f_i(P)=P\otimes\mathbf{R}_i+f_ie_i'(P)[-2]$,
- $e_i'f_{\theta(i)}(P)=P\otimes\mathbf{R}_{\theta(i)}[\lambda_i+\lambda_{\theta(i)}-\nu\cdot(i+\theta(i))/2]+f_{\theta(i)}e_i'(P)[-i\cdot\theta(i)]$,
- $e_i'f_j(P)=f_je_i'(P)[-i\cdot j]$ if $i\neq j,\theta(j)$.

Note that Lemma 8.32 implies these relations up to some filtration. Hence, since the associated graded is projective, they hold in full generality. This proves the first claim of part (a) of the theorem. Next, recall the following fact, see [EK1, prop. 2.5], [EK3, prop. 2.11].

8.33. Claim. *There is a* ${}^\theta\mathbf{B}$*-module generated by a non-zero vector* ϕ_λ *such that*

$$e_i\phi_\lambda=0,\quad t_i\phi_\lambda=v^{\lambda_i+\lambda_{\theta(i)}}\phi_\lambda,\quad \{x;\,e_jx=0,\,\forall j\}=\mathcal{K}\,\phi_\lambda,\quad i\in I.$$

This ${}^\theta\mathbf{B}$*-module is irreducible and it is unique up to an isomorphism.*

So we must check that the ${}^\theta\mathbf{B}$-module ${}^\theta\mathbf{V}(\lambda)$ satisfies the axioms above. It is generated by ϕ_λ by Lemma 8.34 below. The other axioms are obvious.

Part (b) of the theorem follows from [EK2, prop. 4.2(ii)] and Lemma 8.9(b). The bilinear form $(\bullet:\bullet)$ is the same as the bilinear form obtained from (8.2) by base change from $\mathcal{B}$ to $\mathcal{K}$.

Finally, for part (c) of the theorem it is enough to check that $(f_iP)^\sharp=f_i(P^\sharp)$ for any graded module P in ${}^\theta$**R-proj**. By Lemma 8.34 below we may assume that $P={}^\theta\mathbf{R}_\mathbf{y}$ for some $\mathbf{y}$. By (8.5) we can also assume that $\mathbf{y}=\mathbf{i}$ with $\mathbf{i}\in{}^\theta I^\nu$. Then the claim follows from the formulas in Proposition 8.14, because ${}^\theta\mathbf{R}_\mathbf{i}$ is $\sharp$-selfdual for any $\mathbf{i}$, see Section 8.10.

∎

8.34. Lemma. *Any object of* ${}^\theta\mathbf{R}_m$-**proj** *is of the form* ${}^\theta\mathbf{R}_m \otimes_{\mathbf{R}_m} P$ *for some* P *in* $\mathbf{R}_m$-**proj**. *The* $\mathcal{A}$*-module* ${}^\theta\mathbf{K}_{I,m}$ *is spanned by the* ${}^\theta\mathbf{R}_{\mathbf{y}}$*'s with* $\mathbf{y}$ *in* ${}^\theta Y^m$.

Proof : Let b be a simple object in ${}^\theta\mathbf{R}_\nu$-$\mathbf{fMod}_0$ with $|\nu| = m$. We'll view it as an element of ${}^\theta B(\lambda)$. An easy induction using Proposition 8.23 implies that for each integer $a \geqslant 1$ we have

$$f_i^{(a)}\,{}^\theta G^{\mathrm{low}}(b) = \left\langle \begin{matrix} \varepsilon_i(b)+a \\ a \end{matrix} \right\rangle {}^\theta G^{\mathrm{low}}(\tilde F_i^a b) + \sum_{b'} f_{b,b'}\,{}^\theta G^{\mathrm{low}}(b'),$$

where b' runs over the set of elements of ${}^\theta B(\lambda)$ such that $\varepsilon_i(b') > \varepsilon_i(b)+a$ and $f_{b,b'}$ lies in $\mathcal{A}$. Therefore, for any i such that $\varepsilon_i(b) \geqslant 1$, we have

$$f_i^{(\varepsilon_i(b))}\,{}^\theta G^{\mathrm{low}}(\tilde E_i^{(\varepsilon_i(b))} b) = {}^\theta G^{\mathrm{low}}(b) + \sum_{b'} f_{b,b'}\,{}^\theta G^{\mathrm{low}}(b')$$

by Proposition 8.22(c), (e). Here b' runs over the set of elements of ${}^\theta B(\lambda)$ such that $\varepsilon_i(b') > \varepsilon_i(b)$. Thus a simultaneous induction on ν and descending induction on $\varepsilon_i(b)$ implies that ${}^\theta G^{\mathrm{low}}(b)$ lies in the $\mathcal{A}$-span of the elements $f_{\mathbf{y}}(\mathbf{k})$ with $\mathbf{y} \in Y^m$. We are done, because $f_{\mathbf{y}}(\mathbf{k}) = {}^\theta\mathbf{R}_{\theta(\mathbf{y})\mathbf{y}}$, see Section 8.10.

$\square$

8.35. Remark. The ${}^\theta\mathbf{B}$-module ${}^\theta\mathbf{V}(\lambda)$ is the same as the ${}^\theta\mathbf{B}$-module $V_\theta(\lambda+\theta(\lambda))$ in [EK1, prop. 2.5]. Let $(\bullet : \bullet)_{KE}$ be the bilinear form on ${}^\theta\mathbf{V}(\lambda)$ considered in loc. cit. We have

$$(P : Q) = (1-v^2)^{-m}(P : Q)_{KE}, \quad \forall P, Q \in {}^\theta\mathbf{R}_m\text{-}\mathbf{proj}.$$

Note that $(\bullet : \bullet)_{KE}$ is a symmetric $\mathcal{A}$-bilinear form ${}^\theta\mathbf{K}_I \times {}^\theta\mathbf{K}_I \to \mathcal{A}$, and that Theorem 8.31$(b)$ yields

$$(e_i x : y)_{KE} = (x : f_i y)_{KE}, \quad i \in I,\ x, y \in {}^\theta\mathbf{V}(\lambda).$$

8.36. Results over an arbitrary field k. Recall that $p, q \in \mathbf{k}^\times$ and that I is a $\mathbb{Z} \rtimes \mathbb{Z}_2$-invariant subset of $\mathbf{k}^\times$. We associate to I a quiver with an involution (Γ, θ) as in Section 6.2. Fix $\lambda \in \mathbb{N}I$ as in (6.2), i.e., we set

$$\lambda = \sum_i i, \quad i \in I \cap \{q, -q\}.$$

The graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}_m$, defined in Sections 5.1, 6.4, and the operators e_i, f_i on ${}^\theta\mathbf{K}_I$, defined in (8.4), make sense over any field $\mathbf{k}$ (not necessarily algebraically closed nor of characteristic zero). For any $\mathbf{k}$ there is again a ${}^\theta\mathbf{B}$-module isomorphism

$${}^\theta\mathbf{V}(\lambda) = \mathcal{K} \otimes_{\mathcal{A}} {}^\theta\mathbf{K}_I,$$

where ${}^\theta\mathbf{V}(\lambda)$ is the Enomoto-Kashiwara's ${}^\theta\mathbf{B}$-module. To prove this it is enough to check the axioms in Claim 8.33. The proof is the same as in characteristic zero. Note that the $\mathcal{K}$-algebra ${}^\theta\mathbf{B}$ and the ${}^\theta\mathbf{B}$-module ${}^\theta\mathbf{V}(\lambda)$ depend only on (Γ, θ) (i.e., on I and p), and on λ (i.e., on q). Therefore, for each m, the number of simple graded ${}^\theta\mathbf{R}_m$-modules is the same for any field $\mathbf{k}$ as long as Γ, θ, λ remain unchanged. In particular all simple graded ${}^\theta\mathbf{R}_m$-modules are absolutely irreducible. Recall that the simple graded ${}^\theta\mathbf{R}_m$-modules are finite dimensional, because ${}^\theta\mathbf{R}_m$ is finitely generated over its center. Therefore all simple graded ${}^\theta\mathbf{R}_m$-modules are *split simple*, i.e., with a one dimensional endomorphism $\mathbf{k}$-algebra, see e.g, [L,thm. 7.5]. Note that, for $\mathbf{k}$ is algebraically closed, we already use this when claiming that the Cartan pairing is perfect.

The discussion above, Theorem 6.5, and Remark 6.10 imply that the number of simple graded $\mathbf{H}_m$-modules in $\mathbf{Mod}_I$ is the same for any field $\mathbf{k}$ of characteristic $\neq 2$ as long as Γ, θ, λ remain unchanged.

9. Presentation of the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector ν in ${}^\theta\mathbb{N}I$ and a dimension vector λ in $\mathbb{N}I$. Set $|\nu| = 2m$. Fix an object $(\mathbf{V}, \varpi)$ in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ and an object $\mathbf{\Lambda}$ in $\boldsymbol{\mathcal{V}}_\lambda$. In this section we give a proof of Theorem 5.8. By Theorem 4.17 and Corollary 5.6 there is a unique injective graded $\mathbf{k}$-algebra homomorphism

$$\Phi : {}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} \to {}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}},$$

$$1_{\mathbf{i}} \mapsto 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, \quad \varkappa_{\mathbf{i},l} \mapsto \varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l), \quad \sigma_{\mathbf{i},k} \mapsto \sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(k), \quad \pi_{\mathbf{i},1} \mapsto \pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(1),$$

$$\mathbf{i} \in {}^\theta I^\nu, \quad k = 1, \dots, m-1, \quad l = 1, 2, \dots, m.$$

We must prove that Φ is a surjective map. Note that both algebras have 1, because the set ${}^\theta I^\nu$ is finite. Since the grading does not matter, we can replace ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$ by ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$. To unburden the notation we abbreviate

$${}^\theta\mathbf{R}_\nu = {}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}, \quad {}^\theta\mathbf{F}_\nu = {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}, \quad {}^\theta\mathbf{Z}_\nu = {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}},$$

$$\varkappa_{\nu,\mathbf{i}}(l) = \varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l), \quad \sigma_{\nu,\mathbf{i}}(k) = \sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(k), \quad \pi_{\nu,\mathbf{i}}(1) = \pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(1),$$

$${}^\theta\mathcal{Z}_\nu = {}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}, \quad {}^\theta Z_\nu = {}^\theta Z_{\mathbf{\Lambda},\mathbf{V}}, \quad \textit{etc.}$$

9.1. The filtration of ${}^\theta\mathbf{Z}_\nu$. Recall that W_m is regarded as a Coxeter group of type B_m with the set of simple reflections $\{s_1, s_2, \dots, s_m\}$. From now on let $\leqslant$ and ℓ be the corresponding Bruhat order and length function. For a future use we set also

$$\Delta^+ = \Delta^+_s \sqcup \Delta^+_l, \quad \Delta^+_s = \{\chi_k \pm \chi_l;\, 1 \leqslant l < k \leqslant m\}, \quad \Delta^+_l = \{\chi_1, \chi_2, \dots, \chi_m\}.$$

Note that $\leqslant$, ℓ, and Δ^+ differ from the order, the length function, and the set of positive roots introduced in Section 4.2. We hope this will not create any confusion. We can now introduce a filtration of ${}^\theta Z_\nu$ by closed subsets. We define

$${}^\theta O^{\leqslant x}_\nu = \bigcup_{w \leqslant x} {}^\theta O^w_\nu, \quad {}^\theta Z^{\leqslant x}_\nu = q^{-1}({}^\theta O^{\leqslant x}_\nu), \quad {}^\theta\mathcal{Z}^{\leqslant x}_\nu = H^{{}^\theta G_\nu}_*({}^\theta Z^{\leqslant x}_\nu, \mathbf{k}).$$

9.2. Lemma. *(a) The set ${}^\theta Z_\nu^{\leqslant x}$ is closed. The variety ${}^\theta Z_\nu^x$ is smooth if $\ell(x)=1$.*

(b) The direct image by the inclusion ${}^\theta Z_\nu^{\leqslant x}\subset {}^\theta Z_\nu$ is an injection ${}^\theta\mathcal{Z}_\nu^{\leqslant x}\subset {}^\theta\mathcal{Z}_\nu$.

(c) The convolution product maps ${}^\theta\mathcal{Z}_\nu^{\leqslant x}\times{}^\theta\mathcal{Z}_\nu^{\leqslant y}$ into ${}^\theta\mathcal{Z}_\nu^{\leqslant xy}$ for each x, y such that $\ell(xy)=\ell(x)+\ell(y)$.

(d) The unit of ${}^\theta\mathcal{Z}_\nu$ lies in ${}^\theta\mathcal{Z}_\nu^e$.

Proof: To avoid confusions, let $\leqslant_D$ and ℓ_D be the Bruhat order and length function introduced in Section 4.2. The claims in the lemma are well-known if we replace $\leqslant$, ℓ by $\leqslant_D$, ℓ_D respectively. Therefore, it is enough to prove the following : assume that ${}^\theta O^v_{\nu,x,y}$ and ${}^\theta O^w_{\nu,x,y}$ are non empty. Then ${}^\theta O^v_{\nu,x,y}\subset{}^\theta\bar O^w_{\nu,x,y}$ iff $v\leqslant w$. Up to the action of a well-chosen diagonal element we may assume that $x=e$. We can also assume that y is minimal in the coset $W_\nu y$. Since ${}^\theta O^v_{\nu,e,y}$ and ${}^\theta O^w_{\nu,e,y}$ are non empty, we have $v,w\in W_\nu y$. Finally, on the coset $W_\nu y$ the orders $\leqslant$, $\leqslant_D$ are the same because $W_\nu\subset\mathfrak{S}_m$, see (4.2) and the last remark in Section 4.2.

$\square$

Let ${}^\theta\mathbf{Z}_\nu^{\leqslant x}$ be the image of ${}^\theta\mathcal{Z}_\nu^{\leqslant x}$ by the isomorphism ${}^\theta\mathbf{Z}_\nu={}^\theta\mathcal{Z}_\nu$ in Proposition 3.1(b).

9.3. PBW theorem for ${}^\theta\mathbf{Z}_\nu$. Recall that

$$ {}^\theta\mathbf{F}_\nu=\bigoplus_{\mathbf{i}\in{}^\theta I^\nu}\mathbf{k}[x_\mathbf{i}(1),x_\mathbf{i}(2),\dots,x_\mathbf{i}(m)], $$

see Section 4.11. The graded $\mathbf{k}$-algebra ${}^\theta\mathbf{Z}_\nu$ has a natural structure of graded ${}^\theta\mathbf{F}_\nu$-module such that $x_\mathbf{i}(l)$ acts as $\varkappa_{\nu,\mathbf{i}}(l)$ under the inclusion ${}^\theta\mathbf{Z}_\nu\subset\mathrm{End}({}^\theta\mathbf{F}_\nu)$ in Theorem 4.17. Recall that ${}^\theta\mathbf{Z}_\nu={}^\theta\mathcal{Z}_\nu$. The following is immediate, see e.g., [CG, sec. 5.5].

9.4. Lemma. *We have ${}^\theta\mathbf{Z}_\nu^{\leqslant x}=\bigoplus_{w\leqslant x}{}^\theta\mathbf{F}_\nu\,[{}^\theta Z_\nu^w]$ for each x. In particular ${}^\theta\mathbf{Z}_\nu$ is a free graded ${}^\theta\mathbf{F}_\nu$-module of rank $2^m m!$.*

The map Φ is a graded ${}^\theta\mathbf{F}_\nu$-module homomorphism. For each x we set

$$ {}^\theta\mathbf{R}_\nu^{\leqslant x}=\sum_{w\leqslant x}{}^\theta\mathbf{F}_\nu\,1_\nu\sigma_{\dot w}, $$

where $\sigma_{\dot w}$ is defined as in (5.2). It is a graded ${}^\theta\mathbf{F}_\nu$-submodule of ${}^\theta\mathbf{R}_\nu$. We abbreviate ${}^\theta\mathbf{R}_\nu^e={}^\theta\mathbf{R}_\nu^{\leqslant e}$. The proof of the surjectivity of Φ consists of two steps. First we prove that $\Phi({}^\theta\mathbf{R}_\nu^{\leqslant x})\subset{}^\theta\mathbf{Z}_\nu^{\leqslant x}$. Then we prove that this inclusion is an equality.

9.5. Step 1. Since Φ is a ${}^\theta\mathbf{F}_\nu$-module homomorphism it is enough to prove that the element $\Phi(\sigma_{\dot x})$ lies in ${}^\theta\mathbf{Z}_\nu^{\leqslant x}$. By an easy induction on the length of x it is enough to observe that we have

$$ \Phi(1)\in{}^\theta\mathbf{Z}_\nu^e,\quad \Phi(\sigma_k)\subset{}^\theta\mathbf{Z}_\nu^{\leqslant s_k},\quad k=1,2,\dots,m. $$

This follows from the definition of the elements $\sigma_\nu(k)$, $\pi_\nu(1)$ of ${}^\theta\mathbf{Z}_\nu$ in Section 4.14. Recall that $s_m=\varepsilon_1$ and $\sigma_m=\pi_1$, see (5.2) for details.

9.6. Step 2. Note that ${}^\theta\mathbf{Z}^e_\nu$ is the free ${}^\theta\mathbf{F}_\nu$-module of rank one generated by $[{}^\theta Z^e_\nu]$. Therefore we have

$$\Phi({}^\theta\mathbf{R}^e_\nu) = {}^\theta\mathbf{F}_\nu\,[{}^\theta Z^e_\nu] = {}^\theta\mathbf{Z}^e_\nu.$$

To complete the proof of Step 2 we are reduced to prove the following.

9.7. Lemma. *If $\ell(s_k w) = \ell(w) + 1$ and $k = 1, 2, \dots m$, then we have the following formula in ${}^\theta\mathbf{Z}^{\leqslant s_k w}_\nu / {}^\theta\mathbf{Z}^{< s_k w}_\nu$:*

$$[{}^\theta Z^{s_k}_\nu] \star [{}^\theta Z^w_\nu] = [{}^\theta Z^{s_k w}_\nu].$$

Proof : By Lemmas 9.2(c) and 9.4 there is an unique element c in ${}^\theta\mathbf{F}_\nu$ such that

$$[{}^\theta Z^{s_k}_\nu] \star [{}^\theta Z^w_\nu] = c \star [{}^\theta Z^{s_k w}_\nu] \text{ in } {}^\theta\mathbf{Z}^{\leqslant s_k w}_\nu / {}^\theta\mathbf{Z}^{< s_k w}_\nu.$$

Let us prove that $c = 1$. For each x, y, z there is a unique element $\Lambda^x_{y,z}$ in $\mathbf{Q}$ such that

$$[{}^\theta Z^x_\nu] = \sum_{y,z} \Lambda^x_{y,z} \psi_{y,z},$$

see Section 4.12. Since $\phi_{\nu,y,yx}$ is a smooth point of ${}^\theta Z^x_\nu$ we have also

$$\Lambda^x_{y,yx} = \mathrm{eu}({}^\theta Z^x_\nu, \phi_{\nu,y,yx})^{-1}.$$

Hence, in the expansion of the element $[{}^\theta Z^{s_k w}_\nu]$ in the $\mathbf{Q}$-basis $(\psi_{y,z})$ the coordinate along the vector $\psi_{x,xs_k w}$ is equal to

$$\Lambda^{s_k w}_{x,xs_k w} = \mathrm{eu}({}^\theta Z^{s_k w}_\nu, \phi_{\nu,x,xs_k w})^{-1}.$$

On the other hand, since $\Lambda^w_{x,xs_k w} = 0$ and

$$[{}^\theta Z^{s_k}_\nu] = \sum_x (\Lambda^{s_k}_{x,x} \psi_{x,x} + \Lambda^{s_k}_{x,xs_k} \psi_{x,xs_k}),$$

the coordinate of $[{}^\theta Z^{s_k}_\nu] \star [{}^\theta Z^w_\nu]$ along $\psi_{x,xs_k w}$ is equal to

$$\Lambda^{s_k}_{x,xs_k} \Lambda^w_{xs_k,xs_k w} \Lambda_{xs_k} = \mathrm{eu}({}^\theta Z^{s_k}_\nu, \phi_{\nu,x,xs_k})^{-1} \mathrm{eu}({}^\theta Z^w_\nu, \phi_{\nu,xs_k,xs_k w})^{-1} \Lambda_{xs_k}.$$

Thus we must check that

$$\mathrm{eu}({}^\theta Z^{s_k}_\nu, \phi_{\nu,x,xs_k})\,\mathrm{eu}({}^\theta Z^w_\nu, \phi_{\nu,xs_k,xs_k w}) = \mathrm{eu}({}^\theta Z^{s_k w}_\nu, \phi_{\nu,x,xs_k w})\,\Lambda_{xs_k}.$$

This follows from the lemma below.

9.8. Lemma. *(a) For $x, y \in W$ we have*

$$\mathrm{eu}({}^\theta O^y_\nu, \phi_{\nu,x,xy}) = \mathrm{eu}({}^\theta\mathfrak{n}_{\nu,x} \oplus {}^\theta\mathfrak{m}_{\nu,xy,x}),$$

$$\mathrm{eu}({}^\theta Z^y_\nu, \phi_{\nu,x,xy}) = \mathrm{eu}({}^\theta O^y_\nu, \phi_{\nu,x,xy})\,\mathrm{eu}({}^\theta\mathfrak{e}^*_{\nu,x,xy}),$$

$$\Lambda_x = \mathrm{eu}({}^\theta Z^e_\nu, \phi_{\nu,x,x}) = \mathrm{eu}({}^\theta F_\nu, \phi_{\nu,x})\,\mathrm{eu}({}^\theta\mathfrak{e}^*_{\nu,x}).$$

(b) For $w, x, y \in W$ such that $\ell(xy) = \ell(x) + \ell(y)$ we have

$$\mathrm{eu}({}^\theta O_\nu^{xy}, \phi_{\nu,w,wxy})\,\mathrm{eu}({}^\theta F_\nu, \phi_{\nu,wx}) = \mathrm{eu}({}^\theta O_\nu^{x}, \phi_{\nu,w,wx})\,\mathrm{eu}({}^\theta O_\nu^{y}, \phi_{\nu,wx,wxy}),$$

$$\mathrm{eu}({}^\theta\mathfrak{e}^*_{\nu,w,wxy} \oplus {}^\theta\mathfrak{e}^*_{\nu,wx}) = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\nu,w,wx} \oplus {}^\theta\mathfrak{e}^*_{\nu,wx,wxy}).$$

Proof : Part (a) is left to the reader. Compare Proposition 4.13 where similar formulas are given. Let us prove (b). Clearly we can assume $w = e$. Set

$$\Delta(y)^- = y(\Delta^+) \cap \Delta^-, \quad \Delta(y)^+ = y(\Delta^-) \cap \Delta^+.$$

Let the symbol $\sqcup$ denote a disjoint union. Recall that

$$\ell(xy) = \ell(x) + \ell(y) \Rightarrow \begin{cases} \Delta(xy)^- = \Delta(x)^- \sqcup x(\Delta(y)^-), \\ \Delta(xy)^+ = \Delta(x)^+ \sqcup x(\Delta(y)^+). \end{cases} \tag{9.1}$$

For $x, y \in W$ the T-module ${}^\theta\mathfrak{m}_{\nu,xy,x}$ is the sum of the root subspaces whose weight belong to the set $x(\Delta(y)^-) \cap {}^\theta\Delta_\nu$, and the T-module ${}^\theta\mathfrak{n}_{\nu,x}$ is the sum of the root subspaces whose weight belong to the set $x(\Delta^+) \cap {}^\theta\Delta_\nu$, see Sections 4.8, 4.9 for details. Thus, by (a), the first claim follows from the following equality

$$\Delta^+ \sqcup \Delta(xy)^- \sqcup x(\Delta^+) = \Delta^+ \sqcup \Delta(x)^- \sqcup x(\Delta^+) \sqcup x(\Delta(y)^-).$$

This equality is a consequence of the first identity in (9.1). Now, let us concentrate on the second claim. Set

$$S_{x,xy} = x(\Delta_s^+ \cap y(\Delta_s^+)), \quad L_{x,xy} = x(\Delta_l^+ \cap y(\Delta_l^+)).$$

There are integers $h_\alpha \geqslant 0$ such that the character of the T-modules ${}^\theta E_{\mathbf{V}}$, ${}^\theta\mathfrak{e}_{\nu,x}$ and ${}^\theta\mathfrak{e}_{\nu,x,xy}$ are of the following form

$$\mathrm{ch}({}^\theta E_{\mathbf{V}}) = \sum_\alpha h_\alpha\,\alpha, \quad \alpha \in \Delta,$$

$$\mathrm{ch}({}^\theta\mathfrak{e}_{\nu,x}) = \sum_\alpha h_\alpha\alpha + \sum_l \lambda_{i_l}\chi_l, \quad \alpha \in x(\Delta_s^+), \quad \chi_l \in x(\Delta_l^+),$$

$$\mathrm{ch}({}^\theta\mathfrak{e}_{\nu,x,xy}) = \sum_\alpha h_\alpha\alpha + \sum_l \lambda_{i_l}\chi_l, \quad \alpha \in S_{x,xy}, \quad \chi_l \in L_{x,xy}.$$

See Section 4.9 for details. Let

$$S = S_{e,xy} \sqcup x(\Delta_s^+), \quad S' = S_{e,x} \sqcup S_{x,xy},$$

$$L = L_{e,xy} \sqcup x(\Delta_l^+), \quad L' = L_{e,x} \sqcup L_{x,xy}.$$

We obtain

$$\mathrm{ch}({}^\theta\mathfrak{e}_{\nu,e,xy} \oplus {}^\theta\mathfrak{e}_{\nu,x}) = \sum_\alpha h_\alpha\alpha + \sum_l \lambda_{i_l}\chi_l, \quad \alpha \in S, \quad \chi_l \in L,$$

$$\mathrm{ch}({}^\theta\mathfrak{e}_{\nu,e,x} \oplus {}^\theta\mathfrak{e}_{\nu,x,xy}) = \sum_\alpha h_\alpha\alpha + \sum_l \lambda_{i_l}\chi_l, \quad \alpha \in S', \quad \chi_l \in L'.$$

Now, a short computation yields

$$S=S' \iff \Delta_s^+\cap\Delta(xy)^+=\Delta_s^+\cap\big(\Delta(x)^+\sqcup x(\Delta(y)^+)\big),$$

$$L=L' \iff \Delta_l^+\cap\Delta(xy)^+=\Delta_l^+\cap\big(\Delta(x)^+\sqcup x(\Delta(y)^+)\big).$$

Thus the claim follows from (9.1). □

10. Perverse sheaves on ${}^\theta\mathbf{E}_{\mathbf{\Lambda},\mathbf{V}}$ and the global bases of ${}^\theta\mathbf{V}(\lambda)$

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector ν in ${}^\theta\mathbb{N}I$ and a dimension vector λ in $\mathbb{N}I$. Set $|\nu|=2m$. Fix an object $(\mathbf{V},\varpi)$ in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ and an object $\mathbf{\Lambda}$ in $\boldsymbol{\mathcal{V}}_\lambda$. To unburden the notation we'll abbreviate

$${}^\theta G_\nu={}^\theta G_{\mathbf{V}},\quad {}^\theta\mathbf{R}_\nu={}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}.$$

10.1. Perverse sheaves on ${}^\theta E_\nu$. First, we generalize the setting in Section 2. We define *an orientation* Ω *of* I to be a partition $I=\Omega\sqcup\bar\Omega$. Fix an orientation Ω. For each I-graded $\mathbb{C}$-vector space $\mathbf{W}$ we write $\mathbf{W}_\Omega=\bigoplus_{i\in\Omega}\mathbf{W}_i$. Now, we define

$$L_{\mathbf{\Lambda},\mathbf{V},\Omega}=\mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\mathbf{\Lambda}_\Omega,\mathbf{V}_\Omega)\oplus\mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\mathbf{V}_{\bar\Omega},\mathbf{\Lambda}_{\bar\Omega}),\quad {}^\theta E_{\mathbf{\Lambda},\mathbf{V},\Omega}={}^\theta E_{\mathbf{V}}\times L_{\mathbf{\Lambda},\mathbf{V},\Omega}.$$

An element of ${}^\theta E_{\mathbf{\Lambda},\mathbf{V},\Omega}$ is a triplet (x,y,z) with

$$x\in{}^\theta E_{\mathbf{V}},\quad y\in\mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\mathbf{\Lambda}_\Omega,\mathbf{V}_\Omega),\quad z\in\mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\mathbf{V}_{\bar\Omega},\mathbf{\Lambda}_{\bar\Omega}).$$

For each $\mathbf{y}$ in ${}^\theta Y^\nu$ we define also

$${}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y},\Omega}=\{(x,y,z,\phi)\in{}^\theta E_{\mathbf{\Lambda},\mathbf{V},\Omega}\times{}^\theta F_{\mathbf{V},\mathbf{y}}\,;\,\phi=(\mathbf{V}^l),\,x(\mathbf{V}^l)\subset\mathbf{V}^l,$$
$$y(\mathbf{\Lambda})\subset\mathbf{V}^0,\,z(\mathbf{V}^0)=0\}.$$

To unburden the notation we'll abbreviate

$${}^\theta E_{\nu,\Omega}={}^\theta E_{\mathbf{\Lambda},\mathbf{V},\Omega},\quad {}^\theta F_{\mathbf{y}}={}^\theta F_{\mathbf{V},\mathbf{y}},\quad {}^\theta\widetilde{F}_{\mathbf{y},\Omega}={}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y},\Omega}.$$

We define the semisimple complex ${}^\theta\mathcal{L}_{\mathbf{y},\Omega}$ over ${}^\theta E_{\nu,\Omega}$ as the direct image of the constant sheaf over ${}^\theta\widetilde{F}_{\mathbf{y},\Omega}$ by the obvious projection. We define ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}$ as the set of isomorphism classes of simple perverse sheaves over ${}^\theta E_{\nu,\Omega}$ which appear as a direct factor of ${}^\theta\mathcal{L}_{\mathbf{y},\Omega}[d]$ for some $\mathbf{y}\in{}^\theta Y^\nu$ and $d\in\mathbb{Z}$. Next, we define ${}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega}$ as the full subcategory of $\boldsymbol{\mathcal{D}}_{{}^\theta G_{\mathbf{V}}}({}^\theta E_{\nu,\Omega})$ consisting of the objects which are isomorphic to finite direct sums of $\mathcal{L}[d]$ with $\mathcal{L}\in{}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}$ and $d\in\mathbb{Z}$. When there is no risk of confusion we abbreviate

$${}^\theta\boldsymbol{\mathcal{P}}={}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega},\quad {}^\theta\boldsymbol{\mathcal{Q}}={}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega},\quad {}^\theta\mathcal{L}_{\mathbf{y}}={}^\theta\mathcal{L}_{\mathbf{y},\Omega}.$$

10.2. Example. Let Γ, θ, and λ be as in Sections 6.2, 6.4. Set $\bar{\Omega} = \emptyset$, and $\nu = i + \theta(i)$ for some $i \in I$. We have ${}^\theta E_{\nu,\Omega} = L_i \times L_{\theta(i)}$ with $L_j = \mathrm{Hom}(\mathbf{\Lambda}_j, \mathbf{V}_j)$, ${}^\theta I^\nu = \{\mathbf{i}, \theta(\mathbf{i})\}$ with $\mathbf{i} = i\theta(i)$, and

$$ {}^\theta\widetilde{F}_{\mathbf{i},\Omega} = \{(\mathbf{V} \supset \mathbf{V}_{\theta(i)} \supset 0)\} \times L_{\theta(i)}, \quad {}^\theta\widetilde{F}_{\theta(\mathbf{i}),\Omega} = \{(\mathbf{V} \supset \mathbf{V}_i \supset 0)\} \times L_i. $$

Therefore the following holds

- if $\lambda_i + \lambda_{\theta(i)} \neq 0$ then ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega} = \{\mathbf{k}_{L_i}[\lambda_i], \mathbf{k}_{L_{\theta(i)}}[\lambda_{\theta(i)}]\}$, ${}^\theta\mathcal{L}^\delta_{\theta(\mathbf{i})} = \mathbf{k}_{L_i}[\lambda_i]$, and ${}^\theta\mathcal{L}^\delta_{\mathbf{i}} = \mathbf{k}_{L_{\theta(i)}}[\lambda_{\theta(i)}]$,
- if $\lambda_i + \lambda_{\theta(i)} = 0$ then ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega} = \{\mathbf{k}_{\{0\}}\}$ and ${}^\theta\mathcal{L}^\delta_{\theta(\mathbf{i})} = {}^\theta\mathcal{L}^\delta_{\mathbf{i}} = \mathbf{k}_{\{0\}}$.

10.3. Multiplication of complexes. Set $\nu'' = \nu + \nu' + \theta(\nu')$ with $\nu' \in \mathbb{N}I$. Fix $\mathbf{V}' \in \boldsymbol{\mathcal{V}}_{\nu'}$ and $\mathbf{V}'' \in {}^\theta\boldsymbol{\mathcal{V}}_{\nu''}$. Let T be the set of triples (V, γ, γ') where

- V is an I-graded subspace of $\mathbf{V}''$ such that $\mathbf{V}''/V \in \boldsymbol{\mathcal{V}}_{\nu'}$ and $V^\perp \subset V$,
- $\gamma : \mathbf{V} \to V/V^\perp$ is an isomorphism in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$,
- $\gamma' : \mathbf{V}' \to \mathbf{V}''/V$ is an isomorphism in $\boldsymbol{\mathcal{V}}_{\nu'}$.

We consider the following diagram

$$ {}^\theta E_{\nu,\Omega} \times E_{\nu'} \xleftarrow{p_1} {}^\theta E_{1,\Omega} \xrightarrow{p_2} {}^\theta E_{2,\Omega} \xrightarrow{p_3} {}^\theta E_{\nu'',\Omega}. $$

Here ${}^\theta E_{2,\Omega}$ is the variety of tuples (x, y, z, V) where

- V is an I-graded subspace of $\mathbf{V}''$ such that $\mathbf{V}''/V \in \boldsymbol{\mathcal{V}}_{\nu'}$ and $V^\perp \subset V$,
- $(x, y, z) \in {}^\theta E_{\nu'',\Omega}$, $y(\mathbf{\Lambda}) \subset V$, $x(V) \subset V$, and $z(V^\perp) = 0$,

and ${}^\theta E_{1,\Omega}$ is the variety of tuples $(x, y, z, V, \gamma, \gamma')$ where

- $(V, \gamma, \gamma') \in T$,
- $(x, y, z, V) \in {}^\theta E_{2,\Omega}$.

Finally the maps are given by

- $p_1(x, y, z, V, \gamma, \gamma') = (x_\gamma, y_\gamma, z_\gamma, x_{\gamma'})$,
- $p_2(x, y, z, V, \gamma, \gamma') = (x, y, z, V)$,
- $p_3(x, y, z, V) = (x, y, z)$,

where

- $x_\gamma = \gamma^{-1} \circ (x|_{V/V^\perp}) \circ \gamma$,
- $x_{\gamma'} = (\gamma')^{-1} \circ (x|_{\mathbf{V}''/V}) \circ \gamma'$,
- $y_\gamma = \gamma^{-1} \circ y$.
- $z_\gamma = z \circ \gamma$.

The group ${}^\theta G_{\nu''}$ acts on ${}^\theta E_{1,\Omega}$, ${}^\theta E_{2,\Omega}$ and the maps p_2, p_3 are ${}^\theta G_{\nu''}$-equivariant. Note that p_1 is a smooth map with connected fibers, that p_2 is a principal bundle, and that p_3 is proper. Therefore, for any complexes $\mathcal{E} \in \boldsymbol{\mathcal{D}}_{{}^\theta G_\nu}({}^\theta E_{\nu,\Omega})$ and $\mathcal{E}' \in \boldsymbol{\mathcal{D}}_{G_{\nu'}}(E_{\nu'})$ there is a unique complex $\mathcal{E}_2 \in \boldsymbol{\mathcal{D}}_{{}^\theta G_{\nu''}}({}^\theta E_{2,\Omega})$ such that

$$ p_1^*(\mathcal{E} \boxtimes \mathcal{E}') = p_2^*(\mathcal{E}_2). $$

Then, we define a complex $\mathcal{E}'' = \varphi_!(\mathcal{E}, \mathcal{E}')$ in $\boldsymbol{\mathcal{D}}_{{}^\theta G_{\nu''}}({}^\theta E_{\nu'',\Omega})$ by

$$ \mathcal{E}'' = (p_3)_!(\mathcal{E}_2). $$

Now, let $\nu' = i$. Hence $E_{\nu'} = 0$. Let $\mathcal{L}_i = \mathbf{k}_{E_{\nu'}}$, the trivial complex over $E_{\nu'}$.

10.4. Definition. *Set $\nu' = i$. For $\mathcal{E}$ in $\boldsymbol{\mathcal{D}}_{{}^\theta G_\nu}({}^\theta E_{\nu,\Omega})$ we define the complex*

$$\underline{f_i}(\mathcal{E}) = \varphi_!(\mathcal{E}, \mathcal{L}_i)[b_{\nu,i}], \quad b_{\nu,i} = \nu_i + \sum_j \nu_j\, h_{i,j} + \lambda_{\Omega,\theta(i)} + \lambda_{\bar\Omega,i}.$$

10.5. Proposition. *(a) $\underline{f_i}$ yields a functor ${}^\theta\boldsymbol{\mathcal{Q}} \to {}^\theta\boldsymbol{\mathcal{Q}}$.*

(b) $\underline{f_i}({}^\theta\mathcal{L}^\delta_{\mathbf{i}}) = {}^\theta\mathcal{L}^\delta_{i\mathbf{i}\theta(i)}$ for each $\mathbf{i}$ in ${}^\theta I^\nu$.

Proof : A standard computation yields

$$\varphi_!({}^\theta\mathcal{L}_{\mathbf{y}}, \mathcal{L}_i) = {}^\theta\mathcal{L}_{i\mathbf{y}\theta(i)}, \quad \mathbf{y} \in {}^\theta Y^\nu.$$

See [E, prop. 4.11], [L2, sec. 9.2.6-7]. This implies (a). The same computation as in Proposition 2.5 yields

$$d_{\mathbf{i}} = \ell_\nu/2 + \sum_{k<l;\, k+l\neq 1} h_{i_k,i_l}/2 + \sum_{1\leqslant l} (\lambda_{\Omega,i_l} + \lambda_{\bar\Omega,\theta(i_l)}), \tag{10.1}$$

where $\lambda_{\Omega,i} = \lambda_i$ if $i \in \Omega$ and 0 else. Thus we have

$$d_{\lambda,i\mathbf{i}\theta(i)} - d_{\lambda,\mathbf{i}} = b_{\nu,i}.$$

Part (b) follows from this equality.

$\square$

10.6. Restriction of complexes. Set $\nu'' = \nu + \nu' + \theta(\nu')$ with $\nu' \in \mathbb{N}I$. Fix $\mathbf{V}' \in \boldsymbol{\mathcal{V}}_{\nu'}$, $\mathbf{V}'' \in {}^\theta\boldsymbol{\mathcal{V}}_{\nu''}$, and fix a triple $(V, \gamma, \gamma') \in T$. Consider the diagram

$${}^\theta E_{\nu,\Omega} \times E_{\nu'} \xleftarrow{\kappa} {}^\theta E_{3,\Omega} \xrightarrow{\iota} {}^\theta E_{\nu'',\Omega}.$$

Here we have set

- ${}^\theta E_{3,\Omega} = \{(x,y,z) \in {}^\theta E_{\nu'',\Omega}\,;\, x(V) \subset V,\ y(\boldsymbol{\Lambda}) \subset V,\ z(V^\perp) = 0\}$,
- $\kappa(x,y,z) = (x_\gamma, y_\gamma, z_\gamma, x_{\gamma'})$,
- $\iota(x,y,z) = (x,y,z)$.

For any $\mathcal{E}''$ in $\boldsymbol{\mathcal{D}}_{{}^\theta G_{\nu''}}({}^\theta E_{\nu'',\Omega})$ we define a complex in $\boldsymbol{\mathcal{D}}_{{}^\theta G_\nu \times G_{\nu'}}({}^\theta E_{\nu,\Omega} \times E_{\nu'})$ by

$$\varphi^*(\mathcal{E}'') = \kappa_!\iota^*(\mathcal{E}'').$$

10.7. Definition. *Set $\nu' = i$. For any $\mathcal{E}''$ in $\boldsymbol{\mathcal{D}}_{{}^\theta G_{\nu''}}({}^\theta E_{\nu'',\Omega})$ we define*

$$\underline{e_i}(\mathcal{E}'') = \varphi^*(\mathcal{E}'')[a_{\nu,i}], \quad a_{\nu,i} = -\nu_i + \sum_j \nu_j\, h_{i,j} + \lambda_{\Omega,\theta(i)} + \lambda_{\bar\Omega,i}.$$

10.8. Proposition. *(a)* $\underline{e_i}$ *yields a functor from* ${}^\theta\mathcal{Q} \to {}^\theta\mathcal{Q}$.

(b) $\underline{e_i}({}^\theta\mathcal{L}_{\mathbf{y}}) = \bigoplus_k {}^\theta\mathcal{L}_{\mathbf{y}_k}[-2d_k]$ *for some integer* d_k. *The sum runs over all* k *such that* $i_k = i$, *and*

$$\mathbf{y}_k = (\mathbf{i}, \mathbf{a}^{(k)}), \quad \mathbf{a}^{(k)} = (a_l^{(k)}), \quad a_l^{(k)} = a_l - \delta_{l,k} - \delta_{l,1-k}.$$

(c) $\underline{e_i}({}^\theta\mathcal{L}_{\mathbf{i}}^\delta) = \bigoplus_{\mathbf{i}'} {}^\theta\mathcal{L}_{\mathbf{i}'}^\delta[\deg(\mathbf{i}', \theta(i); \mathbf{i})]$, *where* $\mathbf{i}'$ *runs over all sequences such that* $\mathbf{i}$ *lies in* $Sh(\mathbf{i}', \theta(i))$.

Proof: Part (a) follows from (b). Parts (b) and (c) are analogues of [E, prop. 4.11(ii)] (which itself is an analogue of [L2, sec. 9.2.6]), where the case $\lambda = 0$ is considered. Our proof is similar. Assume that $\nu' = i$. Hence we have $E_{\nu'} = \{0\}$. We define

$${}^\theta\widetilde{F}_{3,\Omega} = \{(x, y, z, \phi) \in {}^\theta\widetilde{F}_{\mathbf{y},\Omega}\,;\, (x, y, z) \in {}^\theta E_{3,\Omega}\},$$

$${}^\theta F_3^{(k)} = \{\phi = (\mathbf{V}^l) \in {}^\theta F_{\mathbf{y}}\,;\, \mathbf{V}^k \subset V,\, \mathbf{V}^{k-1} \not\subset V\},$$

$${}^\theta\widetilde{F}_{3,\Omega}^{(k)} = \{(x, y, z, \phi) \in {}^\theta\widetilde{F}_{3,\Omega};\, \phi \in {}^\theta F_3^{(k)}\}.$$

Note that ${}^\theta\widetilde{F}_{3,\Omega} = \bigcup_k {}^\theta\widetilde{F}_{3,\Omega}^{(k)}$ is a partition into locally closed subsets. Let $\mathbf{y}_k$ be as above. Consider the map

$$f_k : {}^\theta\widetilde{F}_{3,\Omega}^{(k)} \to {}^\theta\widetilde{F}_{\mathbf{y}_k,\Omega}, \quad (x, y, z, \phi) \mapsto (x_\gamma, y_\gamma, z_\gamma, \phi_\gamma),$$

where ϕ_γ is the flag whose l-th term is equal to

$$\gamma^{-1}\big((V \cap \mathbf{V}^l + V^\perp)/V^\perp\big).$$

We get the following diagram, whose right square is Cartesian

$$\begin{tikzcd}
{}^\theta\widetilde{F}_{\mathbf{y}_k,\Omega} \arrow[d] & {}^\theta\widetilde{F}_{3,\Omega}^{(k)} \arrow[l, "f_k"'] \arrow[r] & {}^\theta\widetilde{F}_{3,\Omega} \arrow[r] \arrow[d] & {}^\theta\widetilde{F}_{\mathbf{y},\Omega} \arrow[d] \\
{}^\theta E_{\nu,\Omega} & & {}^\theta E_{3,\Omega} \arrow[ll, "\kappa"'] \arrow[r, "\iota"] & {}^\theta E_{\nu'',\Omega}.
\end{tikzcd}$$

It is easy to prove that f_k is an affine bundle. Let $d_k = d_{f_k}$ be its relative dimension. A standard argument using the diagram above yields

$$\varphi^*({}^\theta\mathcal{L}_{\mathbf{y}}) = \kappa_!\iota^*({}^\theta\mathcal{L}_{\mathbf{y}}) = \bigoplus_{i_k = i} {}^\theta\mathcal{L}_{\mathbf{y}_k}[-2d_k].$$

This proves (b). Now, we concentrate on (c). Assume that $\mathbf{y} = \mathbf{i}$ lies in ${}^\theta I^{\nu''}$. Therefore we have

$$\mathbf{i} = (i_{-m}, i_{1-m}, \dots, i_{m+1}), \quad k = -m, 1-m, \dots, m+1.$$

First, we compute explicitly the integer d_k. The map ${}^\theta F_3^{(k)} \to {}^\theta F_{\mathbf{y}_k}$, $\phi \mapsto \phi_\gamma$ is an affine bundle of relative dimension

$$\sharp\{l\,;\, -m \leqslant l < k,\, i_l = i\}.$$

Further, for each tuple (x,y,z,ϕ) in ${}^\theta\widetilde{F}_{\mathbf{i}_k,\Omega}$ and for each $\phi'\in{}^\theta F_3^{(k)}$such that $\phi'_\gamma=\phi$, the space of tuples (x',y',z') in ${}^\theta E_{3,\Omega}$ such that (x',y',z',ϕ') lies in ${}^\theta\widetilde{F}_{3,\Omega}$ and $\kappa(x',y',z')=(x,y,z)$ has the dimension

$$\sum_{k<l\leqslant m+1} h_{i,i_l}+\delta_{k\leqslant 0}\,(\lambda_{\Omega,\theta(i)}+\lambda_{\bar\Omega,i}-h_{i,\theta(i)}).$$

See [E, prop. 4.11(ii)] for details. Therefore we have

$$d_k=\sum_{k<l\leqslant m+1} h_{i,i_l}+\delta_{k\leqslant 0}\,(\lambda_{\Omega,\theta(i)}+\lambda_{\bar\Omega,i}-h_{i,\theta(i)})+\sharp\{l\,;\,-m\leqslant l<k,\,i_l=i\}.$$

Next, (10.1) yields

$$d_{\lambda,\mathbf{i}}-d_{\lambda,\mathbf{i}_k}=\nu_i+\sum_{-m\leqslant l<k} h_{i_l,i}+\sum_{k<l\leqslant m+1} h_{i,i_l}+\delta_{k\geqslant 1}(\lambda_{\Omega,i}+\lambda_{\bar\Omega,\theta(i)}-h_{\theta(i),i})+$$
$$+\delta_{k\leqslant 0}(\lambda_{\bar\Omega,i}+\lambda_{\Omega,\theta(i)}-h_{i,\theta(i)}).$$

Finally we have

$$a_{\nu,i}=-\nu_i+\sum_{-m\leqslant l\leqslant m+1} h_{i,i_l}-h_{i,\theta(i)}+\lambda_{\Omega,\theta(i)}+\lambda_{\bar\Omega,i}.$$

Therefore we get

$$a_{\nu,i}+d_{\lambda,\mathbf{i}}-d_{\lambda,\mathbf{i}_k}-2d_k=-\sum_{-m\leqslant l<k} i\cdot i_l+\delta_{k\geqslant 1}(i\cdot\theta(i)+\lambda_i+\lambda_{\theta(i)}).$$

On the other hand $\deg(\mathbf{i}_k,\theta(i)\,;\mathbf{i})$ is the degree of the homogeneous element $\sigma_{\dot w}1_{\mathbf{i}}$, where $\dot w$ is a reduced decomposition of an element w of W_{m+1} such that $w(\mathbf{i})=i\,\mathbf{i}_k\theta(i)$. If $k\leqslant 0$ then we can choose $\dot w=s_ms_{m-1}\dots s_{1-k}$ and we get

$$\deg(\mathbf{i}_k,\theta(i)\,;\mathbf{i})=-\sum_{-m\leqslant l<k} i\cdot i_l.$$

If $k\geqslant 1$ then we can choose $\dot w=s_ms_{m-1}\dots s_1\varepsilon_1s_1\dots s_{k-1}$ and we get

$$\begin{aligned}\deg(\mathbf{i}_k,\theta(i)\,;\mathbf{i})&=-\sum_{-m\leqslant l\leqslant 0} i\cdot i_l+i\cdot\theta(i)+\lambda_i+\lambda_{\theta(i)}-\sum_{1\leqslant l<k} i\cdot i_l,\\ &=-\sum_{-m\leqslant l<k} i\cdot i_l+i\cdot\theta(i)+\lambda_i+\lambda_{\theta(i)}.\end{aligned}$$

The proposition is proved.

□

10.9. Example. Let Γ, θ, λ, ν, and Ω be as in Example 10.2. Let $\mathbf{k}$ denote the unique element of ${}^\theta\boldsymbol{\mathcal{P}}_{0,\Omega}$. We have

$$\{0\}\xleftarrow{\kappa}L_{\theta(i)}\xrightarrow{\iota}L_i\times L_{\theta(i)}\;.$$

We obtain

$$a_{\nu,i}=\lambda_{\theta(i)},\quad \underline{e_i}(\mathbf{k}_{L_i}[\lambda_i])=\mathbf{k}[\lambda_i+\lambda_{\theta(i)}],\quad \underline{e_i}(\mathbf{k}_{L_{\theta(i)}}[\lambda_{\theta(i)}])=\mathbf{k},$$
$$b_{\nu,i}=\lambda_{\theta(i)},\quad \underline{f_i}(\mathbf{k})=\mathbf{k}_{L_{\theta(i)}}[\lambda_{\theta(i)}].$$

10.10. A key estimate. First, let us introduce the following notation. For any complex of constructible sheaves $\mathcal{L}$ and any integer d we'll write $v^d\mathcal{L}$ for the shifted complex $\mathcal{L}[d]$.

10.11. Proposition. *For each $i \in I$ there are maps*

$$\underline{\varepsilon}_i : {}^\theta\boldsymbol{\mathcal{P}} \cup \{0\} \to \mathbb{Z}_{\geqslant 0}, \quad \underline{\tilde{E}}_i : {}^\theta\boldsymbol{\mathcal{P}} \to {}^\theta\boldsymbol{\mathcal{P}} \cup \{0\}, \quad \underline{\tilde{F}}_i : {}^\theta\boldsymbol{\mathcal{P}} \to {}^\theta\boldsymbol{\mathcal{P}}$$

such that $\underline{\varepsilon}_i(0) = 0$, and for each $\mathcal{L}$ in ${}^\theta\boldsymbol{\mathcal{P}}$ the following hold

(a) we have $\underline{\varepsilon}_i(\underline{\tilde{E}}_i(\mathcal{L})) = \underline{\varepsilon}_i(\mathcal{L}) - 1$ and

$$\underline{e_i}(\mathcal{L}) = v^{1-\underline{\varepsilon}_i(\mathcal{L})}\, \underline{\tilde{E}}_i(\mathcal{L}) + \sum_{\mathcal{L}'} e_{\mathcal{L},\mathcal{L}'}\mathcal{L}',$$

$$\mathcal{L}' \in {}^\theta\boldsymbol{\mathcal{P}}, \quad \underline{\varepsilon}_i(\mathcal{L}') \geqslant \underline{\varepsilon}_i(\mathcal{L}), \quad e_{\mathcal{L},\mathcal{L}'} \in v^{1-\underline{\varepsilon}_i(\mathcal{L}')}\mathbb{Z}[v],$$

(b) we have $\underline{\varepsilon}_i(\underline{\tilde{F}}_i(\mathcal{L})) = \underline{\varepsilon}_i(\mathcal{L}) + 1$ and

$$\underline{f_i}(\mathcal{L}) = \langle \underline{\varepsilon}_i(\mathcal{L}) + 1 \rangle\, \underline{\tilde{F}}_i(\mathcal{L}) + \sum_{\mathcal{L}'} f_{\mathcal{L},\mathcal{L}'}\mathcal{L}',$$

$$\mathcal{L}' \in {}^\theta\boldsymbol{\mathcal{P}}, \quad \underline{\varepsilon}_i(\mathcal{L}') > \underline{\varepsilon}_i(\mathcal{L}) + 1, \quad f_{\mathcal{L},\mathcal{L}'} \in v^{2-\underline{\varepsilon}_i(\mathcal{L}')}\mathbb{Z}[v],$$

(c) we have

$$\underline{\varepsilon}_i(0) = 0, \quad \underline{\tilde{E}}_i(\mathcal{L}) \neq 0 \text{ if } \underline{\varepsilon}_i(\mathcal{L}) > 0,$$

$$\underline{\tilde{E}}_i\underline{\tilde{F}}_i(\mathcal{L}) = \mathcal{L}, \quad \underline{\tilde{F}}_i\underline{\tilde{E}}_i(\mathcal{L}) = \mathcal{L} \text{ if } \underline{\tilde{E}}_i(\mathcal{L}) \neq 0,$$

(d) if $\mathcal{L} \in {}^\theta\boldsymbol{\mathcal{P}}$ is such that $\underline{\varepsilon}_i(\mathcal{L}) = 0$ for all i, then $\mathcal{L} \in {}^\theta\boldsymbol{\mathcal{P}}_{0,\Omega}$,

(e) the elements of ${}^\theta\boldsymbol{\mathcal{P}}$ are selfdual.

Proof : We'll prove the proposition for any quiver $\Gamma = (I, H)$ with an involution θ such that Γ has no 1-loops and θ has no fixed points. The estimates in (a), (b) are analogue of [E, thm. 5.3], where they are proved under the assumption $\lambda = 0$. Our proof is essentially the same. Fix a vertex i. First, we can assume that

- i is a sink of Γ,
- $i \in \Omega$,
- $\theta(i) \in \bar{\Omega}$.

More precisely we have the following lemma. Its proof is left to the reader. It is proved as in [E, thm. 4.19], [L2], using Fourier transforms.

10.12. Lemma. *Let $(\Gamma^{(1)}, \theta^{(1)})$, $(\Gamma^{(2)}, \theta^{(2)})$ be two quivers with involutions without fixed points. Assume that $\Gamma^{(1)}$, $\Gamma^{(2)}$ have the set of vertices I and that they have the same set of unoriented arrows. Let $\Omega^{(1)}$, $\Omega^{(2)}$ be two orientations of I. There is an equivalence of categories ${}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega^{(1)}} \to {}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega^{(2)}}$ which commutes with the functors $\underline{f_i}$, $\underline{e_i}$ and with the Verdier duality. The categories ${}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega^{(1)}}$ and ${}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega^{(2)}}$ are relative to the quivers $\Gamma^{(1)}$, $\Gamma^{(2)}$ respectively. This equivalence yields a bijection ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega^{(1)}} \to {}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega^{(2)}}$.*

Then, for each integer $r \geqslant 0$ let ${}^\theta E_{\nu,\Omega,\geqslant r}$ be the closed subset of ${}^\theta E_{\nu,\Omega}$ consisting of the triples (x, y, z) such that there is a I-graded subspace $W \subset \mathbf{V}$ of codimension vector ri such that

$$x(W) \subset W, \quad y(\mathbf{\Lambda}) \subset W, \quad z(W^\perp) = 0.$$

Then, we set

$${}^\theta E_{\nu,\Omega,r} = {}^\theta E_{\nu,\Omega,\geqslant r} \setminus {}^\theta E_{\nu,\Omega,\geqslant r+1}, \quad {}^\theta E_{\nu,\Omega,\leqslant r} = {}^\theta E_{\nu,\Omega} \setminus {}^\theta E_{\nu,\Omega,\geqslant r+1}.$$

Finally we set $\underline{\varepsilon}_i(0) = 0$ and for $\mathcal{L} \in {}^\theta\boldsymbol{\mathcal{P}}$ we define

$$\underline{\varepsilon}_i(\mathcal{L}) = \max\{r\,;\, \sup(\mathcal{L}) \subset {}^\theta E_{\nu,\Omega,\geqslant r}\}.$$

Set $\nu' = i$ and consider the diagram

$${}^\theta E_{\nu,\Omega} \xleftarrow{p_1} {}^\theta E_{1,\Omega} \xrightarrow{p_2} {}^\theta E_{2,\Omega} \xrightarrow{p_3} {}^\theta E_{\nu'',\Omega}.$$

Under restriction it yields the diagram

$${}^\theta E_{\nu,\Omega,r} \longleftarrow {}^\theta E_{1,\Omega,r} \longrightarrow {}^\theta E_{2,\Omega,r+1} \longrightarrow {}^\theta E_{\nu'',\Omega,r+1},$$

where

$${}^\theta E_{1,\Omega,r} = p_1^{-1}({}^\theta E_{\nu,\Omega,r}), \quad {}^\theta E_{2,\Omega,r+1} = p_3^{-1}({}^\theta E_{\nu'',\Omega,r+1}).$$

Note that we have ${}^\theta E_{1,r} = p_2^{-1}({}^\theta E_{2,r+1})$ and that the map ${}^\theta E_{2,r+1} \to {}^\theta E_{\nu'',r+1}$ is a $\mathbb{P}^r$-bundle. Finally, we set $p = p_3 p_2$ and we define ${}^\theta E_{2,\Omega,\leqslant r}$ and ${}^\theta E_{1,\Omega,\leqslant r}$ in the obvious way.

Now, we concentrate on (b). Fix a simple perverse sheaf $\mathcal{L} \in {}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}$. Set $\varepsilon = \underline{\varepsilon}_i(\mathcal{L})$. The maps p_1, p_2 are smooth with connected fibers of dimension d_{p_1}, d_{p_2} such that

$$b_{\nu,i} = d_{p_1} - d_{p_2}. \tag{10.2}$$

Thus, there is a unique simple ${}^\theta G_{\nu''}$-equivariant perverse sheaf $\mathcal{L}_2$ on ${}^\theta E_{2,\Omega}$ with

$$p_1^*(\mathcal{L})[b_{\nu,i}] = p_2^*(\mathcal{L}_2).$$

We have

$$\underline{f_i}(\mathcal{L}) = (p_3)_!(\mathcal{L}_2).$$

The complex $\underline{f_i}(\mathcal{L})$ is supported on ${}^\theta E_{\nu'',\Omega,\varepsilon+1}$. Further, the restriction of $\mathcal{L}_2$ to ${}^\theta E_{2,\Omega,\leqslant\varepsilon+1}$ is supported on ${}^\theta E_{2,\Omega,\varepsilon+1}$, and it is constant along the fibers of p_3 by ${}^\theta G_{\nu''}$-equivariance. Thus

$$\mathcal{L}_2|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon+1}} = p_3^*(\mathcal{L}'')[\varepsilon]$$

for some simple perverse sheaf $\mathcal{L}''$ on ${}^\theta E_{\nu'',\Omega,\leq\varepsilon+1}$. Let $\mathcal{L}_0$ be the minimal perverse extension of $\mathcal{L}''$ to ${}^\theta E_{\nu'',\Omega}$. Since $\underline{f_i}(\mathcal{L})$ is semi-simple, we get

$$\underline{f_i}(\mathcal{L}) = \langle\varepsilon+1\rangle\,\mathcal{L}_0 + \sum_{\mathcal{L}'} f_{\mathcal{L},\mathcal{L}'}\mathcal{L}',$$

$$\mathcal{L}_0, \mathcal{L}' \in {}^\theta\boldsymbol{\mathcal{P}}_{\nu'',\Omega}, \quad \underline{\varepsilon}_i(\mathcal{L}_0) = \varepsilon+1, \quad \underline{\varepsilon}_i(\mathcal{L}') > \varepsilon+1.$$

Let us that $f_{\mathcal{L},\mathcal{L}'}$ lies in $v^{2-\underline{\varepsilon}_i(\mathcal{L}')}\mathbb{Z}[v]$. Write

$$\underline{f_i}(\mathcal{L}) = \bigoplus_{\mathcal{L}'} \mathcal{L}' \otimes M_{\mathcal{L}'},$$

where $M_{\mathcal{L}'}$ is a complex of $\mathbf{k}$-vector spaces. Set $\varepsilon' = \underline{\varepsilon}_i(\mathcal{L}')$. We have

$$\mathrm{R}\mathcal{H}om(\mathcal{L}', \mathcal{L}')|_{{}^\theta E_{\nu'',\leqslant\varepsilon'}} \otimes M^*_{\mathcal{L}'} \subset \mathrm{R}\mathcal{H}om((p_3)_!(\mathcal{L}_2), \mathcal{L}')|_{{}^\theta E_{\nu'',\leqslant\varepsilon'}}.$$

On the other hand, since p_3 restricts to a $\mathbb{P}^{\varepsilon'-1}$-bundle ${}^\theta E_{2,\Omega,\varepsilon'} \to {}^\theta E_{\nu'',\Omega,\varepsilon'}$, we have

$$\mathrm{R}\mathcal{H}om((p_3)_!(\mathcal{L}_2), \mathcal{L}')|_{{}^\theta E_{\nu'',\leqslant\varepsilon'}} = (p_3)_*\mathrm{R}\mathcal{H}om(\mathcal{L}_2, \mathcal{L}'[\varepsilon'-1])|_{{}^\theta E_{2,\leqslant\varepsilon'}}[\varepsilon'-1].$$

Since $\mathcal{L}'[\varepsilon'-1]|_{{}^\theta E_{2,\leqslant\varepsilon'}}$ is a perverse sheaf the complex

$$\mathrm{R}\mathcal{H}om(\mathcal{L}_2, \mathcal{L}'[\varepsilon'-1])|_{{}^\theta E_{2,\leqslant\varepsilon'}}$$

is concentrated in degrees $\geqslant 0$. Its 0-th cohomology group is zero because $\mathcal{L}_2$ and $\mathcal{L}'[\varepsilon'-1]$ are simple and non isomorphic. Thus the complex

$$\mathrm{R}\mathcal{H}om((p_3)_!(\mathcal{L}_2), \mathcal{L}')|_{{}^\theta E_{\nu'',\leqslant\varepsilon'}}$$

is concentrated in degrees $> 1-\varepsilon'$. This implies the estimate we want.

Next, we prove (a). Fix a triple (V,γ,γ') in T. Observe that the hypothesis on Γ, Ω, i implies that for each (x,y,z,W,ρ,ρ') in ${}^\theta E_{1,\Omega}$ we have $x(W^\perp) = z(W^\perp) = 0$, $x(\mathbf{V}), y(\mathbf{\Lambda}) \subset W$, z is completely determined by its restriction to W, and y is completely determined by its composition with the projection $\mathbf{V} \to \mathbf{V}/W^\perp$. Hence x, y, z are completely determined by x_ρ, y_ρ, z_ρ. Therefore κ is an isomorphism. Consider the diagram

(10.3)
$$\begin{tikzcd} {}^\theta E_{\nu,\Omega} \arrow[r, equal, "\kappa"] & {}^\theta E_{3,\Omega} \arrow[r, "\iota"] \arrow[d, "s"] & {}^\theta E_{\nu'',\Omega} \\ & {}^\theta E_{1,\Omega}, \arrow[ul, "p_1"] \arrow[ur, "p"'] & \end{tikzcd}$$

where

$$\kappa(x,y,z) = (x_\gamma, y_\gamma, z_\gamma), \quad s(x,y,z) = (x,y,z,V,\gamma,\gamma'),$$

$$p_1(x,y,z,W,\rho,\rho') = (x_\rho, y_\rho, z_\rho), \quad p(x,y,z,W,\rho,\rho') = (x,y,z).$$

The left square is Cartesian. Fix a simple perverse sheaf $\mathcal{L}$ in ${}^\theta\boldsymbol{\mathcal{P}}_{\nu'',\Omega}$. Set $\varepsilon = \underline{\varepsilon}_i(\mathcal{L})$. We'll assume that $\varepsilon > 0$ (the case $\varepsilon = 0$ is left to the reader). We have

$$\underline{e_i}(\mathcal{L}) = \kappa_! s^* p^*(\mathcal{L})[a_{\nu,i}].$$

The restriction $\mathcal{L}|_{^\theta E_{\nu'',\Omega,\leqslant\varepsilon}}$ is a simple $^\theta G_{\nu''}$-equivariant perverse sheaf supported on $^\theta E_{\nu'',\Omega,\varepsilon}$. Let d_p, d_s be the relative dimension of the maps p, s. Since p restricts to a smooth map $^\theta E_{1,\Omega,\varepsilon-1} \to {}^\theta E_{\nu'',\Omega,\varepsilon}$, the complex

$$\mathcal{L}_1 = p^*(\mathcal{L})[d_p]|_{^\theta E_{1,\Omega,\leqslant\varepsilon-1}}$$

is again a simple $^\theta G_{\nu''}$-equivariant perverse sheaf. It is constant along the fibers of p_1 by $^\theta G_{\nu''}$-equivariance. Thus

$$\begin{aligned}\mathcal{L}'' &= \kappa_! s^* p^*(\mathcal{L})[d_p + d_s]|_{^\theta E_{\nu,\Omega,\leqslant\varepsilon-1}} \\ &= \underline{e_i}(\mathcal{L})[d_p + d_s - a_{\nu,i}]|_{^\theta E_{\nu,\Omega,\leqslant\varepsilon-1}}\end{aligned}$$

is a simple perverse sheaf over $^\theta E_{\nu,\Omega,\leqslant\varepsilon-1}$. Using (10.2) we get

$$d_p + d_s = d_{p_2} + \varepsilon - 1 - d_{p_1}, \quad d_{p_1} - d_{p_2} = b_{\nu,i}, \quad b_{\nu,i} = \nu_i, \quad a_{\nu,i} = -\nu_i.$$

Therefore, we have

$$d_p + d_s - a_{\nu,i} = \varepsilon - 1.$$

Let $\mathcal{L}_0$ be the minimal perverse extension of $\mathcal{L}''$ to $^\theta E_{\nu,\Omega,}$. Since $\underline{e_i}(\mathcal{L})$ is semi-simple we get

$$\begin{gathered}\underline{e_i}(\mathcal{L}) = v^{1-\varepsilon}\,\mathcal{L}_0 + \sum_{\mathcal{L}'} e_{\mathcal{L},\mathcal{L}'}\mathcal{L}' \\ \mathcal{L}_0, \mathcal{L}' \in {}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}, \quad \underline{\varepsilon}_i(\mathcal{L}_0) = \varepsilon - 1, \quad \underline{\varepsilon}_i(\mathcal{L}') \geqslant \varepsilon.\end{gathered} \tag{10.4}$$

Now, one proves that $e_{\mathcal{L},\mathcal{L}'}$ lies in $v^{1-\underline{\varepsilon}_i(\mathcal{L}')}\mathbb{Z}[v]$ as in [E, thm. 5.3]. More precisely, since $p_1^*\underline{e_i}(\mathcal{L})$ and $p^*(\mathcal{L})[a_{\nu,i}]$ are constant along the fibers of p_1 and since

$$\underline{e_i}(\mathcal{L}) = \kappa_! s^* p^*(\mathcal{L})[a_{\nu,i}],$$

we have

$$p_1^*\underline{e_i}(\mathcal{L}) = p^*(\mathcal{L})[-b_{\nu,i}]. \tag{10.5}$$

On the other hand, we have

$$p_1^*\underline{e_i}(\mathcal{L}) = \bigoplus_{\mathcal{L}''} p_1^*(\mathcal{L}'') \otimes M_{\mathcal{L}''},$$

where the graded $\mathbf{k}$-vector space $M_{\mathcal{L}''}$ is the multiplicity space of the simple perverse sheaf $\mathcal{L}'' \in {}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}$ in $\underline{e_i}(\mathcal{L})$. Let $\mathcal{L}_2''$ be the perverse sheaf over $^\theta E_{2,\Omega}$ such that

$$p_1^*(\mathcal{L}'')[b_{\nu,i}] = p_2^*(\mathcal{L}_2'').$$

We obtain

$$\bigoplus_{\mathcal{L}''} \mathcal{L}_2'' \otimes M_{\mathcal{L}''} = p_3^*(\mathcal{L}).$$

Now, let $\mathcal{L}'$ be as in (10.4). Set $\varepsilon' = \underline{\varepsilon_i}(\mathcal{L}')$. We have

$$\begin{aligned}
&\bigoplus_{\mathcal{L}''}\mathrm{RHom}(\mathcal{L}''_2|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon'+1}}, \mathcal{L}'_2|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon'+1}}) \otimes M^*_{\mathcal{L}''} = \\
&\quad = \mathrm{RHom}(p_3^*(\mathcal{L})|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon'+1}}, \mathcal{L}'_2|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon'+1}}) \\
&\quad = \mathrm{RHom}(\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}}, (p_3)_!(\mathcal{L}'_2)|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}}) \\
&\quad = \mathrm{RHom}(\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}}, \underline{f_i}(\mathcal{L}')|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}}) \\
&\quad = \langle\varepsilon'+1\rangle\,\mathrm{RHom}(\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}}, \underline{\tilde{F}}_i(\mathcal{L}')|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}}),
\end{aligned}$$

where the last equality follows from part (b). The complex

$$\mathrm{RHom}(\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}}, \underline{\tilde{F}}_i(\mathcal{L}')|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}}),$$

is concentrated in degrees $\geqslant 1$, because the perverse sheaves $\mathcal{L}'$ and $\underline{\tilde{F}}_i(\mathcal{L}')$ are simple and distincts. Thus the complex

$$\mathrm{RHom}(\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}}, \underline{f_i}(\mathcal{L}')|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}})$$

is concentrated in degrees $\geqslant 1-\varepsilon'$. Choosing $\mathcal{L}' = \mathcal{L}''$ we get that $M^*_{\mathcal{L}'}$ is also concentrated in degrees $\geqslant 1-\varepsilon'$. Therefore

$$e'_i(\mathcal{L}) = \bigoplus_{\mathcal{L}''}\mathcal{L}''\otimes M_{\mathcal{L}''} = \bigoplus_{\mathcal{L}''}\bigoplus_{d\in\mathbb{Z}} v^{-d}\mathcal{L}''\otimes M_{\mathcal{L}'',d},$$

with $M_{\mathcal{L}'',d} = 0$ unless $-d \geqslant 1-\varepsilon'$. We are done.

Now, we concentrate on (c). The second claim in (c) is obvious. Now, we prove that $\underline{\tilde{E}}_i\underline{\tilde{F}}_i(\mathcal{L}) = \mathcal{L}$ for $\mathcal{L}$ in ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}$. Recall the diagram (10.3). Set $\varepsilon = \underline{\varepsilon_i}(\mathcal{L})$ and take a simple perverse sheaf $\mathcal{L}_2$ on ${}^\theta E_{2,\Omega}$ such that

$$p_1^*(\mathcal{L})[b_{\nu,i}] = p_2^*(\mathcal{L}_2), \quad (p_3)_!(\mathcal{L}_2) = \underline{f_i}(\mathcal{L}).$$

We have

$$(p_3)_!(\mathcal{L}_2)|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon+1}} = \langle\varepsilon+1\rangle\,\underline{\tilde{F}}_i(\mathcal{L})|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon+1}}.$$

On the other hand, since

$$\mathcal{L}_2|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon+1}} = p_3^*(\underline{\tilde{F}}_i\mathcal{L})[\varepsilon]|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon+1}},$$

we have

$$p^*(\underline{\tilde{F}}_i\mathcal{L})|_{{}^\theta E_{1,\Omega,\leqslant\varepsilon+1}} = p_2^*(\mathcal{L}_2)[-\varepsilon]|_{{}^\theta E_{1,\Omega,\leqslant\varepsilon+1}} = p_1^*(\mathcal{L})[b_{\nu,i}-\varepsilon]|_{{}^\theta E_{1,\Omega,\leqslant\varepsilon+1}}.$$

Therefore we have also

$$\begin{aligned}
\underline{e_i}(\underline{\tilde{F}}_i\mathcal{L})|_{{}^\theta E_{\nu,\Omega,\leqslant\varepsilon+1}} &= \kappa_! s^* p^*(\underline{\tilde{F}}_i\mathcal{L})[a_{\nu,i}]|_{{}^\theta E_{\nu,\Omega,\leqslant\varepsilon+1}} \\
&= \mathcal{L}[-\varepsilon]|_{{}^\theta E_{\nu,\Omega,\leqslant\varepsilon+1}}.
\end{aligned}$$

Therefore $\underline{\tilde{E}}_i\underline{\tilde{F}}_i(\mathcal{L}) = \mathcal{L}$. Finally, fix $\mathcal{L} \in {}^\theta\boldsymbol{\mathcal{P}}_{\nu'',\Omega}$ such that $\underline{\varepsilon_i}(\mathcal{L}) > 0$ and let us prove that $\underline{\tilde{F}}_i\underline{\tilde{E}}_i(\mathcal{L}) = \mathcal{L}$. Write $\varepsilon = \underline{\varepsilon_i}(\mathcal{L})$. By (10.5) we have

$$p_1^*\underline{e_i}(\mathcal{L}) = p^*(\mathcal{L})[a_{\nu,i}].$$

Hence we have also

$$p_1^*(\underline{\tilde{E}}_i\mathcal{L})[-a_{\nu,i}]|_{{}^\theta E_{1,\Omega,\leqslant\varepsilon-1}} = p^*(\mathcal{L})[\varepsilon-1]|_{{}^\theta E_{1,\Omega,\leqslant\varepsilon-1}}.$$

Since $p_3^*(\mathcal{L})[\varepsilon-1]|_{{}^\theta E_{2,\Omega,\varepsilon}}$ is a simple perverse sheaf, we have

$$\underline{f_i}(\underline{\tilde{E}}_i\mathcal{L})|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon}} = (p_3)_! p_3^*(\mathcal{L})[\varepsilon-1]|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon}} = \langle\varepsilon\rangle\,\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon}}.$$

This implies that $\underline{\tilde{F}}_i\underline{\tilde{E}}_i(\mathcal{L}) = \mathcal{L}$.

Next, (d) is obvious. If $\nu \neq 0$ we choose $\mathbf{y}$, d such that $\mathcal{L}[d]$ is a direct summand of ${}^\theta\mathcal{L}_{\mathbf{y}}$. We may assume that $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ with $a_1 > 0$. Then $\underline{\varepsilon_{i_1}}(\mathcal{L}) > 0$ by (b) and Proposition 10.5(b).

Finally, we prove (e) by descending induction on ν. Any element in ${}^\theta\boldsymbol{\mathcal{P}}_{0,\Omega}$ is selfdual. Assume that $\nu > 0$. By part (d) there is i such that $\underline{\varepsilon_i}(\mathcal{L}) > 0$. Set $\varepsilon = \underline{\varepsilon_i}(\mathcal{L})$. We prove that $\mathcal{L}$ is selfdual by descending induction on ε. By parts (b), (c) we have

$$\underline{f_i}(\underline{\tilde{E}}_i\mathcal{L}) = \langle\varepsilon\rangle\,\mathcal{L} + \sum_{\mathcal{L}'} f_{\mathcal{L},\mathcal{L}'}\mathcal{L}', \quad \underline{\varepsilon_i}(\mathcal{L}') > \varepsilon.$$

The perverse sheaf $\underline{\tilde{E}}_i\mathcal{L}$ is selfdual by the induction hypothesis on ν. It is easy to see that $\underline{f_i}$ commutes with the Verdier duality. Hence the laft hand side is also selfdual. We have

$$\underline{f_i}(\underline{\tilde{E}}_i\mathcal{L})|_{{}^\theta E_{\nu,\Omega,\leqslant\varepsilon}} = \langle\varepsilon\rangle\,\mathcal{L}|_{{}^\theta E_{\nu,\Omega,\leqslant\varepsilon}}.$$

Since $\mathcal{L}$ is the minimal extension of its restriction to ${}^\theta E_{\nu,\Omega,\leqslant\varepsilon}$, it is selfdual.

□

Let $K({}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega})$ be the Abelian group with one generator $[\mathcal{L}]$ for each isomorphism class of objects of ${}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega}$ and with relations $[\mathcal{L}] + [\mathcal{L}'] = [\mathcal{L}'']$ whenever $\mathcal{L}''$ is isomorphic to $\mathcal{L} \oplus \mathcal{L}'$. To unburden the notation we'll abbreviate

$$K({}^\theta\boldsymbol{\mathcal{Q}}) = \bigoplus_\nu K({}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega}), \quad \mathcal{L} = [\mathcal{L}].$$

Note that $K({}^\theta\boldsymbol{\mathcal{Q}})$ is a free $\mathcal{A}$-module such that $v\mathcal{L} = \mathcal{L}[1]$ and $v^{-1}\mathcal{L} = \mathcal{L}[-1]$. Further the Verdier duality yields an $\mathcal{A}$-antilinear map $K({}^\theta\boldsymbol{\mathcal{Q}}) \to K({}^\theta\boldsymbol{\mathcal{Q}})$.

10.13. Corollary. *The $\mathcal{A}$-module $K({}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega})$ is spanned by $\{{}^\theta\mathcal{L}^\delta_{\mathbf{y}};\, \mathbf{y} \in {}^\theta Y^\nu\}$.*

Proof : The corollary is proved as in Lemma 8.34, using Proposition 10.11 instead of Propositions 8.22, 8.23.

□

10.14. Example. Let Γ, θ, λ, ν be as in Example 10.2, and set $\Omega = \{i\}$. We have ${}^\theta E_{\nu,\Omega} = L_i \times L^*_{\theta(i)}$, ${}^\theta E_{\nu,\Omega,0} = {}^\theta E_{\nu,\Omega} \setminus \{0\}$, and ${}^\theta E_{\nu,\Omega,1} = \{0\}$. We have also

- if $\lambda_i + \lambda_{\theta(i)} \neq 0$ then ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega} = \{\mathbf{k}_{{}^\theta E_{\nu,\Omega}}[\lambda_i + \lambda_{\theta(i)}], \mathbf{k}_{\{0\}}\}$, and

$$\underline{\varepsilon}_i(\mathbf{k}_{{}^\theta E_{\nu,\Omega}}[\lambda_i + \lambda_{\theta(i)}]) = 0, \quad \underline{\varepsilon}_i(\mathbf{k}_{\{0\}}) = 1, \quad \underline{\tilde{F}}_i(\mathbf{k}) = \mathbf{k}_{\{0\}},$$

- if $\lambda_i + \lambda_{\theta(i)} = 0$ then ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega} = \{\mathbf{k}_{\{0\}}\}$, and

$$\underline{\varepsilon}_i(\mathbf{k}_{\{0\}}) = 1, \quad \underline{\tilde{F}}_i(\mathbf{k}) = \mathbf{k}_{\{0\}}.$$

10.15. Comparison of the crystals. We choose Γ, θ and λ as in Sections 6.2, 6.4, and we set $\Omega = I$. We define a functor

$$\mathbf{Y} : {}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega} \to {}^\theta\mathbf{R}_\nu\textbf{-mod}, \quad \mathbf{Y}(\mathcal{L}) = \bigoplus_{\mathbf{i}\in{}^\theta I^\nu} \mathrm{Ext}^*_{{}^\theta G_\nu}({}^\theta\mathcal{L}^\delta_{\mathbf{i}}, \mathcal{L}).$$

The functor $\mathbf{Y}$ is additive and it commutes with the shift (the shift of complexes in the left hand side and the shift of the grading in the right hand side).

10.16. Proposition. *(a)* $\mathbf{Y}$ *takes* ${}^\theta\boldsymbol{\mathcal{Q}}$ *to* ${}^\theta\mathbf{R}$-**proj**, *and* ${}^\theta\mathcal{L}^\delta_{\mathbf{y}}$ *to* ${}^\theta\mathbf{R}_{\mathbf{y}}$. *It maps* ${}^\theta\boldsymbol{\mathcal{P}}$ *bijectively to the set of* $\sharp$*-selfdual indecomposable projective graded modules.*

(b) $\mathbf{Y}$ *yields an* $\mathcal{A}$*-module isomorphism* $K({}^\theta\boldsymbol{\mathcal{Q}}) \to {}^\theta\mathbf{K}_I$ *which maps* ${}^\theta\boldsymbol{\mathcal{P}}$ *bijectively onto* ${}^\theta\mathbf{G}^{\mathrm{low}}(\lambda)$. *It commutes with the duality. We have*

$$e_i \circ \mathbf{Y} = \mathbf{Y} \circ \underline{e_{\theta(i)}}, \quad f_i \circ \mathbf{Y} = \mathbf{Y} \circ \underline{f_{\theta(i)}}.$$

Proof : Theorem 5.8, proved in Section 9, yields a graded $\mathbf{k}$-algebra isomorphism

$${}^\theta\mathbf{R}_\nu = {}^\theta\mathbf{Z}^\delta_\nu.$$

Recall that the right hans side is the graded $\mathbf{k}$-algebra

$${}^\theta\mathbf{Z}^\delta_\nu = \bigoplus_{\mathbf{i},\mathbf{i}'\in{}^\theta I^\nu} \mathrm{Ext}^*_{{}^\theta G_\nu}({}^\theta\mathcal{L}^\delta_{\mathbf{i}}, {}^\theta\mathcal{L}^\delta_{\mathbf{i}'}),$$

equipped with the Yoneda composition, see Sections 2.6, 2.8. Therefore the first claim of (a) is obvious. If the sequence $\mathbf{i}$ of ${}^\theta I^\nu$ is the expansion of the pair $\mathbf{y}$ in ${}^\theta Y^\nu$ then we have

$${}^\theta\mathbf{R}_{\mathbf{i}} = \langle\mathbf{b}\rangle!\,{}^\theta\mathbf{R}_{\mathbf{y}}, \quad {}^\theta\mathcal{L}^\delta_{\mathbf{i}} = \langle\mathbf{b}\rangle!\,{}^\theta\mathcal{L}^\delta_{\mathbf{y}},$$

where $\mathbf{b}$ is a sequence such that the multiplicity of $\mathbf{y}$ is $\theta(\mathbf{b})\mathbf{b}$. See Remark 2.7 and (8.5). Therefore to prove the second claim of (a) it is enough to observe that we have $\mathbf{Y}({}^\theta\mathcal{L}^\delta_{\mathbf{i}}) = {}^\theta\mathbf{R}^\delta_{\mathbf{i}}$. Next, the same proof as in [VV, sec. 4.7] implies that $\mathbf{Y}$ takes any element in ${}^\theta\boldsymbol{\mathcal{P}}$ to an indecomposable projective graded module. Indeed, since $\mathbf{Y}({}^\theta\mathcal{L}^\delta_{\mathbf{y}}) = {}^\theta\mathbf{R}^\delta_{\mathbf{y}}$ and both sides are selfdual, the functor $\mathbf{Y}$ takes the elements of ${}^\theta\boldsymbol{\mathcal{P}}$ to $\sharp$-selfdual indecomposable projective graded modules, see Sections 2.6 and 8.10. Further, any $\sharp$-selfdual indecomposable projective graded module is a direct summand of $\mathbf{Y}({}^\theta\mathcal{L}^\delta_{\mathbf{y}}) = {}^\theta\mathbf{R}^\delta_{\mathbf{y}}$ for some $\mathbf{y}$, hence is the image by $\mathbf{Y}$ of an element of ${}^\theta\boldsymbol{\mathcal{P}}$. Part (a) is proved. Next, the first claim of (b) follows from Definition 8.3, Proposition 8.4(c) and Corollary 10.13. Finally the last claim of (b) follows from Propositions 10.5(b), 10.8(c) and Proposition 8.14.

□

Recall the set ${}^\theta\mathbf{G}^{\mathrm{low}}(\lambda)$ introduced in Definition 8.3. For $b \in {}^\theta B(\lambda)$ let ${}^\theta\mathcal{L}(b)$ denote the unique element in ${}^\theta\boldsymbol{\mathcal{P}}$ such that

$$\mathbf{Y}({}^\theta\mathcal{L}(b)) = {}^\theta G^{\mathrm{low}}(b). \tag{10.6}$$

Hence we have ${}^\theta\boldsymbol{\mathcal{P}} = \{{}^\theta\mathcal{L}(b);\, b \in {}^\theta B(\lambda)\}$. We'll set also ${}^\theta\mathcal{L}(0) = 0$. Combining Propositions 8.23 and 10.11 we can now compare the crystal $({}^\theta B(\lambda), \tilde{E}_i, \tilde{F}_i, \varepsilon_i)$ from Proposition 8.22 with the crystal $({}^\theta\boldsymbol{\mathcal{P}}, \underline{\tilde{E}}_i, \underline{\tilde{F}}_i, \underline{\varepsilon}_i)$ from Proposition 10.11.

10.17. Proposition. *For $i \in I$ and $b \in {}^\theta B(\lambda)$ we have*

$$\underline{\tilde{E}}_i({}^\theta\mathcal{L}(b)) = {}^\theta\mathcal{L}(\tilde{E}_{\theta(i)}b), \quad \underline{\tilde{F}}_i({}^\theta\mathcal{L}(b)) = {}^\theta\mathcal{L}(\tilde{F}_{\theta(i)}b), \quad \underline{\varepsilon}_i({}^\theta\mathcal{L}(b)) = \varepsilon_{\theta(i)}(b).$$

Proof : We can regard $\underline{\varepsilon}_i$, $\underline{\tilde{E}}_i$, and $\underline{\tilde{F}}_i$ as maps

$$\underline{\varepsilon}_i : {}^\theta B(\lambda) \cup \{0\} \to \mathbb{Z}_{\geqslant 0}, \quad \underline{\tilde{E}}_i : {}^\theta B(\lambda) \to {}^\theta B(\lambda) \cup \{0\}, \quad \underline{\tilde{F}}_i : {}^\theta B(\lambda) \to {}^\theta B(\lambda).$$

Propositions 10.11(b), 10.16(b) yield

$$f_{\theta(i)}{}^\theta G^{\mathrm{low}}(b) = \langle \underline{\varepsilon}_i(b) + 1\rangle\, {}^\theta G^{\mathrm{low}}(\underline{\tilde{F}}_i b) + \sum_{b'} f_{b,b'}{}^\theta G^{\mathrm{low}}(b'), \quad \underline{\varepsilon}_i(b') > \underline{\varepsilon}_i(b) + 1.$$

Taking the transpose, Definition 8.8 and Proposition 8.4(a) yield

$$e_{\theta(i)}{}^\theta G^{\mathrm{up}}(b) = \langle \underline{\varepsilon}_i(b)\rangle\, {}^\theta G^{\mathrm{up}}(\underline{\tilde{E}}_i b) + \sum_{b'} f_{b',b}{}^\theta G^{\mathrm{up}}(b'), \quad \underline{\varepsilon}_i(b') < \underline{\varepsilon}_i(b) - 1.$$

Now, recall that

$$\varepsilon_{\theta(i)}(b) = \max\{n \geqslant 0; e^n_{\theta(i)}{}^\theta G^{\mathrm{up}}(b) \neq 0\}, \quad \underline{\varepsilon}_i(\underline{\tilde{E}}_i b) = \underline{\varepsilon}_i(b) - 1.$$

Thus, using Proposition 8.17 and (8.8) we get $\underline{\varepsilon}_i = \varepsilon_{\theta(i)}$. Then, comparing the formulas above with Proposition 8.23 we get $\underline{\tilde{F}}_i = \tilde{F}_{\theta(i)}$. Finally, Proposition 8.22(c) and 10.11(c) yield $\underline{\tilde{E}}_i = \tilde{E}_{\theta(i)}$.

□

10.18. The global bases of ${}^\theta\mathbf{V}(\lambda)$. Since the operators e_i, f_i on ${}^\theta\mathbf{V}(\lambda)$ satisfy the relations $e_i f_i = v^{-2} f_i e_i + 1$, we can define the modified root operators $\tilde{\mathbf{e}}_i$, $\tilde{\mathbf{f}}_i$ on the ${}^\theta\mathbf{B}$-module ${}^\theta\mathbf{V}(\lambda)$ as follows. For $u \in {}^\theta\mathbf{V}(\lambda)$ we write

$$u = \sum_{n \geqslant 0} f_i^{(n)} u_n \text{ with } e_i u_n = 0,$$

$$\tilde{\mathbf{e}}_i(u) = \sum_{n \geqslant 1} f_i^{(n-1)} u_n, \quad \tilde{\mathbf{f}}_i(u) = \sum_{n \geqslant 0} f_i^{(n+1)} u_n.$$

Let $\mathcal{R} \subset \mathcal{K}$ be the set of functions which are regular at $v = 0$. Let ${}^\theta\mathbf{L}(\lambda)$ be the $\mathcal{R}$-submodule of ${}^\theta\mathbf{V}(\lambda)$ spanned by the elements $\tilde{\mathbf{f}}_{i_1} \dots \tilde{\mathbf{f}}_{i_l}(\phi_\lambda)$ with $l \geqslant 0$, $i_1, \dots, i_l \in I$. We can now apply the results in [EK3]. Together with Propositions 10.16 and 10.17 this yields the following, which is the main result of the paper.

10.19. Theorem. *(a) We have*

$$ {}^\theta\mathbf{L}(\lambda) = \bigoplus_{b\in{}^\theta B(\lambda)} \mathcal{R}\,{}^\theta G^{\mathrm{low}}(b), \quad \tilde{\mathbf{e}}_i({}^\theta\mathbf{L}(\lambda)) \subset {}^\theta\mathbf{L}(\lambda), \quad \tilde{\mathbf{f}}_i({}^\theta\mathbf{L}(\lambda)) \subset {}^\theta\mathbf{L}(\lambda), $$

$$ \tilde{\mathbf{e}}_i{}^\theta G^{\mathrm{low}}(b) = {}^\theta G^{\mathrm{low}}(\tilde{E}_i b) \bmod v\,{}^\theta\mathbf{L}(\lambda), \quad \tilde{\mathbf{f}}_i{}^\theta G^{\mathrm{low}}(b) = {}^\theta G^{\mathrm{low}}(\tilde{F}_i b) \bmod v\,{}^\theta\mathbf{L}(\lambda). $$

(b) The assignment $b \mapsto {}^\theta G^{\mathrm{low}}(b) \bmod v\,{}^\theta\mathbf{L}(\lambda)$ yields a bijection from ${}^\theta B(\lambda)$ to the subset of ${}^\theta\mathbf{L}(\lambda)/v{}^\theta\mathbf{L}(\lambda)$ consisting of the $\tilde{\mathbf{f}}_{i_1}\dots\tilde{\mathbf{f}}_{i_l}(\phi_\lambda)$'s. Further ${}^\theta G^{\mathrm{low}}(b)$ is the unique element x of ${}^\theta\mathbf{V}(\lambda)$ satisfying the following conditions

$$ x^\sharp = x, \quad x = {}^\theta G^{\mathrm{low}}(b) \bmod v\,{}^\theta\mathbf{L}(\lambda). $$

(c) For $b, b' \in {}^\theta B(\lambda)$ let $E_{i,b,b'}, F_{i,b,b'} \in \mathcal{A}$ be the coefficients of ${}^\theta G^{\mathrm{low}}(b')$ in $e_{\theta(i)}{}^\theta G^{\mathrm{low}}(b)$, $f_i{}^\theta G^{\mathrm{low}}(b)$ respectively. Then we have

$$ E_{i,b,b'}|_{v=1} = [F_i\Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b')) : \Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b))], $$

$$ F_{i,b,b'}|_{v=1} = [E_i\Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b')) : \Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b))]. $$

Proof : Proposition 10.17 implies that $\mathbf{Y}$ intertwines the crystal operators $\tilde{E}_{\theta(i)}$, $\tilde{F}_{\theta(i)}$ on ${}^\theta B(\lambda)$ and the crystal operators $\underline{\tilde{E}}_i$, $\underline{\tilde{F}}_i$ on ${}^\theta\boldsymbol{\mathcal{P}}$. Proposition 10.16 implies that $\mathbf{Y}$ intertwines the operators $e_{\theta(i)}$, $f_{\theta(i)}$ and $\underline{e}_i$, $\underline{f}_i$. Therefore, formula (10.6) and Proposition 10.11 yield estimate for the action of e_i, f_i on ${}^\theta\mathbf{G}^{\mathrm{low}}(\lambda)$ which were not available in Proposition 8.23. Using these estimates, part (a) follows from [EK3, thm. 4.1, cor. 4.4], [E, Section 2.3]. The first claim in (b) follows from (a) and Proposition 8.22. The second one is obvious. Part (c) follows from Proposition 8.17. More precisely, by Cartan duality we can regard the elements $E_{i,b,b'}$, $F_{i,b,b'}$ of $\mathcal{A}$ as the coefficients of ${}^\theta G^{\mathrm{up}}(b)$ in the expansion of $f_{\theta(i)}{}^\theta G^{\mathrm{up}}(b')$, $e_i{}^\theta G^{\mathrm{up}}(b')$ with respect to the basis ${}^\theta\mathbf{G}^{\mathrm{up}}(\lambda)$. Therefore, by Proposition 8.17 we can regard the integers $E_{i,b,b'}|_{v=1}$, $F_{i,b,b'}|_{v=1}$ as the coefficients of $\Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b))$ in $F_i\Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b'))$, $E_i\Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b'))$ respectively.

□

A. Appendix

The statements above generalizes to affine Hecke algebras of type C.

A.1. Affine Hecke algebras of type C. Fix p, q_0, q_1 in $\mathbf{k}^\times$. For any integer $m \geqslant 0$ we define the affine Hecke algebra $\mathbf{H}_m$ of type C_m to be the affine Hecke algebra of $Sp(2m)$. It admits the following presentation. If $m > 0$ then $\mathbf{H}_m$ is the $\mathbf{k}$-algebra generated by

$$ T_k, \quad X_l^{\pm 1}, \quad k = 0, 1, \dots, m-1, \quad l = 1, 2, \dots, m $$

satisfying the following defining relations :

(a) $X_l X_{l'} = X_{l'} X_l$,

(b) $(T_0T_1)^2 = (T_1T_0)^2$, $T_kT_{k-1}T_k = T_{k-1}T_kT_{k-1}$ if $k \neq 0, 1$, and $T_kT_{k'} = T_{k'}T_k$ if $|k - k'| \neq 1$,

(c) $T_0X_1^{-1} - X_1T_0 = (q_1^{-1} - q_0)X_1 + (q_0q_1^{-1} - 1)$, $T_kX_kT_k = X_{k+1}$ if $k \neq 0$, and $T_kX_l = X_lT_k$ if $l \neq k, k+1$,

(d) $(T_k - p)(T_k + p^{-1}) = 0$ if $k \neq 0$, and $(T_0 - q_0)(T_0 + q_1^{-1}) = 0$.

If $m = 0$ then $\mathbf{H}_0 = \mathbf{k}$, the trivial $\mathbf{k}$-algebra.

A.2. Remark. The affine Hecke algebra of type B_m is equal to $\mathbf{H}_m/(q_0 - q, q_1 - q)$.

A.3. Intertwiners and blocks of $\mathbf{H}_m$. We define

$$\mathbf{A}' = \mathbf{A}[\Sigma^{-1}], \quad \mathbf{H}'_m = \mathbf{A}' \otimes_{\mathbf{A}} \mathbf{H}_m,$$

where Σ is the multiplicative system generated by

$$X_{l'}^{\pm 1} - X_l, \quad X_{l'}^{\pm 1} - p^2X_l, \quad 1 - X_l^2, \quad 1 + q_0X_l^{\pm 1}, \quad 1 - q_1X_l^{\pm 1}, \quad l \neq l'.$$

For $k = 0, \dots, m-1$ the intertwiner φ_k in $\mathbf{H}'_m$ is given by the following formulas

$$\begin{aligned} \varphi_k - 1 &= \frac{X_k - X_{k+1}}{pX_k - p^{-1}X_{k+1}}\,(T_k - p) \qquad \text{if } k \neq 0, \\ \varphi_0 - 1 &= q_1 \frac{X_1^2 - 1}{(X_1 + q_0)(X_1 - q_1)}\,(T_0 - q_0). \end{aligned} \tag{A.1}$$

There is an isomorphism of $\mathbf{A}'$-algebras

$$\mathbf{A}' \rtimes W_m \to \mathbf{H}'_m, \quad s_k \mapsto \varphi_k, \quad \varepsilon_1 \mapsto \varphi_0, \quad k \neq 0.$$

The semi-direct product group $\mathbb{Z} \rtimes \mathbb{Z}_2$ acts on $\mathbf{k}^\times$ as in Section 6.2. Given a $\mathbb{Z} \rtimes \mathbb{Z}_2$-invariant subset I of $\mathbf{k}^\times$ we denote by $\mathbf{H}_m$-$\mathbf{Mod}_I$ the category of all $\mathbf{H}_m$-modules such that the action of $X_1, X_2, \dots, X_m$ is locally finite and all the eigenvalues belong to I. We associate to the set I a quiver Γ with an involution θ as in loc. cit. Finally, we assume that

$$1, -1 \notin I, \quad p, q_0, q_1 \neq 1, -1.$$

Next, we define the element λ of $\mathbb{N}I$ as

$$\lambda = \begin{cases} \sum_i i, & i \in I \cap \{-q_0, q_1\}, \quad \text{if } -q_0 \neq q_1, \\ 2\sum_i i, & i \in I \cap \{q_1\}, \quad \text{if } -q_0 = q_1. \end{cases} \tag{A.2}$$

Finally we define ${}^\theta\mathbf{R}_m$ and ${}^\theta\mathbf{R}_m$-$\mathbf{Mod}_0$ as in Section 6.4. Note that if $q_0 = q_1 = q$ then λ is the same as in (6.2). Fix any formal series $f(\varkappa)$ in $\mathbf{k}[[\varkappa]]$ such that $f(\varkappa) = 1 + \varkappa$ modulo $(\varkappa^2)$.

A.4. Theorem. *There is an equivalence of categories*

$$^\theta\mathbf{R}_m\text{-}\mathbf{Mod}_0 \to \mathbf{H}_m\text{-}\mathbf{Mod}_I, \quad M \mapsto M$$

which is given by

(*a*) X_l *acts on* $1_\mathbf{i}M$ *by* $i_l^{-1}f(\varkappa_l)$ *for* $l = 1, 2, \ldots, m$,

(*b*) T_k *acts on* $1_\mathbf{i}M$ *as follows for* $k = 1, 2, \ldots, m-1$,

$$\begin{aligned}
&\frac{(pf(\varkappa_k) - p^{-1}f(\varkappa_{k+1}))(\varkappa_k - \varkappa_{k+1})}{f(\varkappa_k) - f(\varkappa_{k+1})}\sigma_k + p && \text{if } i_{k+1} = i_k,\\
&\frac{f(\varkappa_k) - f(\varkappa_{k+1})}{(p^{-1}f(\varkappa_k) - pf(\varkappa_{k+1}))(\varkappa_k - \varkappa_{k+1})}\sigma_k + \frac{(p^{-2}-1)f(\varkappa_{k+1})}{pf(\varkappa_k) - p^{-1}f(\varkappa_{k+1})} && \text{if } i_{k+1} = p^2 i_k,\\
&\frac{pi_kf(\varkappa_k) - p^{-1}i_{k+1}f(\varkappa_{k+1})}{i_kf(\varkappa_k) - i_{k+1}f(\varkappa_{k+1})}\sigma_k + \frac{(p^{-1}-p)i_kf(\varkappa_{k+1})}{i_{k+1}f(\varkappa_k) - i_kf(\varkappa_{k+1})} && \text{if } i_{k+1} \neq i_k, p^2 i_k,
\end{aligned}$$

(*c*) T_0 *acts on* $1_\mathbf{i}M$ *as follows*

$$\begin{aligned}
&\frac{(f(\varkappa_1)-1)^2}{(q_1f(\varkappa_1)^2 - q_1^{-1})\varkappa_1^2}\pi_1 + \frac{(q_0^{-1}q_1^{-1}-1)f(\varkappa_1)^2 + 2f(\varkappa_1)}{q_1^{-1}f(\varkappa_1)^2 - q_1} && \text{if } i_1 = -q_0 = q_1,\\
&\frac{(q_1f(\varkappa_1)+q_0)(f(\varkappa_1)-1)}{(1-q_1^2f(\varkappa_1)^2)\varkappa_1}\pi_1 + \frac{(q_0-q_1^{-1})f(\varkappa_1)^2 + (q_1-q_0)f(\varkappa_1)}{f(\varkappa_1)^2 - q_1^2} && \text{if } i_1 = q_1 \neq -q_0,\\
&\frac{(q_0f(\varkappa_1)+q_1)(1-f(\varkappa_1))}{q_1(q_0f(\varkappa_1)^2 - q_0^{-1})\varkappa_1}\pi_1 + \frac{(q_1-q_0^{-1})f(\varkappa_1)^2 - (q_1-q_0)f(\varkappa_1)}{q_1(q_0^{-1}f(\varkappa_1)^2 - q_0)} && \text{if } i_1 = -q_0 \neq q_1,\\
&\frac{(i_1f(\varkappa_1)+q_0)(i_1f(\varkappa_1)-q_1)}{q_1(i_1^2f(\varkappa_1)^2-1)}\pi_1 + \frac{(q_1q_0-1)f(\varkappa_1)^2 + (q_1-q_0)i_1f(\varkappa_1)}{q_1(f(\varkappa_1)^2 - i_1^2)} && \text{if } i_1 \neq -q_0, q_1.
\end{aligned}$$

Proof : Formula $(A.2)$ yields

$$\begin{cases} \lambda_{i_1} = 2 & \text{if } i_1 = -q_0 = q_1,\\ \lambda_{i_1} = 0 & \text{if } i_1 \neq -q_0, q_1,\\ \lambda_{i_1} = 1 & \text{else.} \end{cases}$$

The proof is the same as the proof Theorem 6.5, using (5.4) and (A.1).

□

Now, all the statements in Section 8 generalizes. The proof is straightforward and is left to the reader. In particular, if $^\theta\mathbf{K}_I$ denotes the Grothendieck group of the category $^\theta\mathbf{R}$**-proj**, then we have a canonical isomorphism

$$^\theta\mathbf{V}(\lambda) = \mathcal{K} \otimes_\mathcal{A} {}^\theta\mathbf{K}_I,$$

where $^\theta\mathbf{V}(\lambda)$ the same $^\theta\mathbf{B}$-module as in Theorem 8.31, with λ given by $(A.2)$ instead of (6.2). Theorem 10.19 generalizes as well.

Index of notation

0.1 : $\mathbf{m}$, $\mathfrak{S}_m$, $\mathfrak{S}_{\mathbf{m}}$, ℓ_m, $\ell_{\mathbf{m}}$, $\langle m\rangle$,

0.2 : $K(\mathbf{R})$, $G(\mathbf{R})$, $\hom_{\mathbf{R}}$, $\mathrm{Hom}_{\mathbf{R}}$, $\mathcal{A}$, gdim, $\mathbf{k}$,

0.3 : $\mathbf{S}_G$,

1.1 : Γ, $H_{i,j}$, $h_{i,j}$, $i\to j$, $i\not\to j$, $i\cdot j$, $E_{\mathbf{V}}$, $G_{\mathbf{V}}$, $\boldsymbol{\mathcal{V}}$, Y^ν, I^ν, Y^m,

1.2 : $F_{\mathbf{V},\mathbf{y}}$, $\tilde{F}_{\mathbf{V},\mathbf{y}}$, $\pi_{\mathbf{y}}$, $d_{\mathbf{y}}$,

1.3 : $\mathbf{S}_{\mathbf{V}}$, $\mathcal{L}_{\mathbf{y}}$, $\mathcal{L}^\delta_{\mathbf{y}}$, $\mathbf{Z}_{\mathbf{V}}$, $\mathbf{Z}^\delta_{\mathbf{V}}$, $\mathbf{F}_{\mathbf{V}}$, $\mathbf{R}(\Gamma)_\nu$,

2.1 : θ, ϖ, ${}^\theta\mathbb{N}I$, ${}^\theta E_{\mathbf{V}}$, ${}^\theta E_{\mathbf{\Lambda},\mathbf{V}}$, $L_{\mathbf{\Lambda},\mathbf{V}}$, ${}^\theta G_{\mathbf{V}}$, ${}^\theta\boldsymbol{\mathcal{V}}$,

2.2 : $F(\mathbf{W})$, $F(\mathbf{W},\varpi)$,

2.3 : ${}^\theta I^\nu$, ${}^\theta Y^\nu$, ${}^\theta Y^m$,

2.4 : ${}^\theta F_{\mathbf{V},\mathbf{y}}$, ${}^\theta\tilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y}}$, ${}^\theta\pi_{\mathbf{\Lambda},\mathbf{y}}$, $d_{\lambda,\mathbf{y}}$,

2.6 : ${}^\theta\mathcal{L}_{\mathbf{y}}$, ${}^\theta\mathcal{L}^\delta_{\mathbf{y}}$, ${}^\theta\mathbf{S}_{\mathbf{V}}$, ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$, ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$, $1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$,

2.7 : $\theta(\mathbf{b})\mathbf{b}$,

2.8 : ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$,

3 : ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$, ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$,

4.1 : $G=O(\mathbf{V},\varpi)$, $F=F(\mathbf{V},\varpi)$, T, W, $W_{\mathbf{V}}$,

4.2 : $\phi_{\mathbf{V}}$, $\mathbf{D}_l$, Δ, Δ^+, Π, ${}^\theta B_{\mathbf{V}}$, ${}^\theta\Delta_{\mathbf{V}}$,

4.3 : $\mathfrak{S}_m$, W_m, $w(\mathbf{i})$,

4.4 : $\phi_{\mathbf{V},w}$, $\mathbf{i}_w$, W_ν, ${}^\theta B_{\mathbf{V},w}$, ${}^\theta N_{\mathbf{V},w}$,

4.5 : ${}^\theta O^w_{\mathbf{V}}$, ${}^\theta O^w_{\mathbf{V},x,y}$, ${}^\theta P_{\mathbf{V},x,y}$, ${}^\theta Z^x_{\mathbf{\Lambda},\mathbf{V}}$,

4.7 : $\mathbf{S}$, χ_l, $M[\lambda]$, $\mathrm{eu}(M)$, Λ_w, $\Lambda^x_{w,w'}$,

4.8 : ${}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w}$, ${}^\theta\mathfrak{n}_{\mathbf{\Lambda},\mathbf{V},w}$,

4.9 : ${}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w,w'}$, ${}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w,w'}$, ${}^\theta\mathfrak{n}_{\mathbf{\Lambda},\mathbf{V},w,w'}$, ${}^\theta\mathfrak{m}_{\mathbf{\Lambda},\mathbf{V},w,w'}$,

4.11 : $x_{\mathbf{i}}(l)$,

4.12 : $\mathbf{Q}$, ψ_w, $\psi_{w,w'}$,

4.14 : $\lambda_{\mathbf{i}}(l)$, $h_{\mathbf{i}}(k)$, $\sigma_{\mathbf{\Lambda},\mathbf{V}}(k)$, $\varkappa_{\mathbf{\Lambda},\mathbf{V}}(l)$, $\pi_{\mathbf{\Lambda},\mathbf{V}}(1)$, $\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}(k)$, $\varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}(l)$, $\pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}(1)$,

4.16 : $\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(k)$, $\varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l)$, $\pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(1)$,

5.1 : ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$, $1_{\mathbf{i}}$, σ_k, $\varkappa_l$, π_1, $Q_{i,j}(u,v)$, ω,

5.3 : $\dot{w}$, $\sigma_{\dot{w}}$,

6.1 : $\mathbf{H}_m$, T_k, X_l,

6.2 : φ_k, $\mathbf{H}_m$-$\mathbf{Mod}_I$,

6.4 : ${}^\theta\mathbf{R}_m$, ${}^\theta\mathbf{R}_\nu$, $1_{\nu,\nu'}$, $1_{m,\nu'}$, ${}^\theta\mathbf{R}_m$-$\mathbf{Mod}_0$, ${}^\theta\mathbf{R}_m$-$\mathbf{fMod}_0$, $\mathbf{H}_m$-$\mathbf{fMod}_I$, Ψ,

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